Unit 4 Review

1. a) $f(x) = 3x^5 - 5x^3$

$f'(x) = 15x^4 - 15x^2$
$= 15x^2(x^2-1)$
$= 15x^2(x-1)(x+1)$

f': $+$ | $-$ | $-$ | $+$ at -1, 0, 1

INC: $(-\infty,-1) \cup (1,\infty)$
DEC: $(-1,0) \cup (0,1)$
rel max at $(-1, 2)$
rel min at $(1,-2)$

$f''(x) = 60x^3 - 30x$
$= 30x(2x^2-1)$

f'': $-$ | $+$ | $-$ | $+$ at $-\sqrt{\frac{1}{2}}$, 0, $\sqrt{\frac{1}{2}}$

Concave up: $(-\sqrt{\frac{1}{2}}, 0) \cup (\sqrt{\frac{1}{2}}, \infty)$
concave down: $(-\infty, -\sqrt{\frac{1}{2}}) \cup (0, \sqrt{\frac{1}{2}})$
inflection pts: $(-\sqrt{\frac{1}{2}}, 1.24)$
$(0,0)$
$(\sqrt{\frac{1}{2}}, -1.24)$

b) $f(\theta) = \theta + 2\sin\theta$

$f'(\theta) = 1 + 2\cos\theta$

f': $+$ | $-$ | $+$ at $\frac{2\pi}{3}$, $\frac{4\pi}{3}$

$f''(\theta) = -2\sin\theta$

f'': $-$ | $+$ on 0, π, 2π

INC: $(0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi)$
DEC: $(\frac{2\pi}{3}, \frac{4\pi}{3})$
rel max at $(\frac{2\pi}{3}, 3.83)$
rel min at $(\frac{4\pi}{3}, 2.46)$

concave up: $(\pi, 2\pi)$
concave down: $(0, \pi)$
inflection pts: (π, π)

2. a) $f(x) = (x+2)^{2/3}$ on $[-3, 6]$

$f'(x) = \frac{2}{3}(x+2)^{-1/3}(1) = \frac{2}{3(x+2)^{1/3}}$

f': $-$ | $+$ at -2

$f(-3) = 1$
$f(6) = 4 \rightarrow$ abs. max
$f(-2) = 0 \rightarrow$ abs. min

2. b) $h(x) = \ln(x^2-4)$ on $[-1, 3]$

$$h'(x) = \frac{2x}{x^2-4}$$

$h(-1)$

$h(3) = 1.6$

$h(-2)$

$h(0)$

$h(2)$

$$h' \quad - \quad + \quad - \quad +$$

$$-2 \quad 0 \quad 2$$

3. a) $2x + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0$

$$\frac{dy}{dx}(x-2y) = -2x-y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x-2y} \quad \text{or} \quad \frac{y+2x}{2y-x}$$

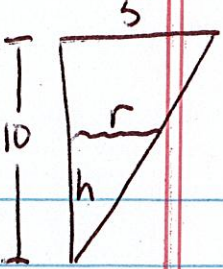
b) $-y\sin x + \cos x\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$

$$\frac{dy}{dx}(\cos x - 2y) = 2x + y\sin x$$

$$\frac{dy}{dx} = \frac{2x + y\sin x}{\cos x - 2y}$$

c) $4\cos x \cdot \cos y\frac{dy}{dx} + (\sin y)(-4\sin x) = 0$

$$\frac{dy}{dx} = \frac{4\sin x\sin y}{4\cos x\cos y} = \tan x\tan y$$

4. 

$\frac{dh}{dt} = \frac{-3}{10}$ cm/hr

$\frac{5}{r} = \frac{10}{h} \Rightarrow 2r = h$

$r = \frac{h}{2}$

a) $V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2(5)$

$V = \frac{125\pi}{12}$ cm^3

b) $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{3}\pi \cdot \frac{h^3}{4}$

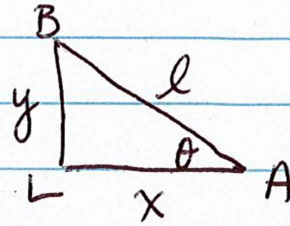
$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{\pi}{4}(5)^2\left(-\frac{3}{10}\right)$

$\frac{dV}{dt} = -\frac{15\pi}{8}$ cm^3/hr

c) Omit, for now...

5. $\frac{dx}{dt} = -15$ km/hr $\quad \frac{dy}{dt} = 10$ km/hr



a) $x^2 + y^2 = l^2$

$4^2 + 3^2 = l^2$

$5 = l$

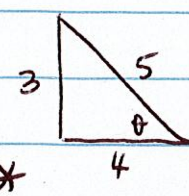
b) $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2l\frac{dl}{dt}$

$2(4)(-15) + 2(3)(10) = 2(5)\frac{dl}{dt}$

-6 km/hr $= \frac{dl}{dt}$

c) $\sin\theta = \frac{y}{l}$

* $\cos\theta \frac{d\theta}{dt} = \frac{l\frac{dy}{dt} - y\frac{dl}{dt}}{l^2}$



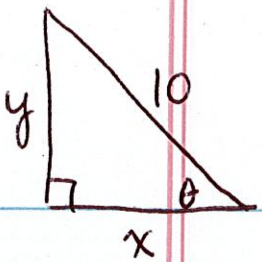
* $\cos\theta = \frac{4}{5}$

$\frac{4}{5}\frac{d\theta}{dt} = \frac{5(10) - (3)(-6)}{25}$

$\frac{4}{5}\frac{d\theta}{dt} = \frac{68}{25}$

$\frac{d\theta}{dt} = \frac{17}{5}$ radians/hr

6.



$\frac{dy}{dt} = -2$ ft/sec

$x^2 + y^2 = 100$

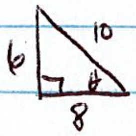
when $y = 6$, $x = 8$
by Pythagorean Thm.

a) $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(8)\frac{dx}{dt} + 2(6)(-2) = 0$

$\frac{dx}{dt} = \frac{3}{2}$ ft/sec

b) $\sin\theta = \frac{y}{10}$



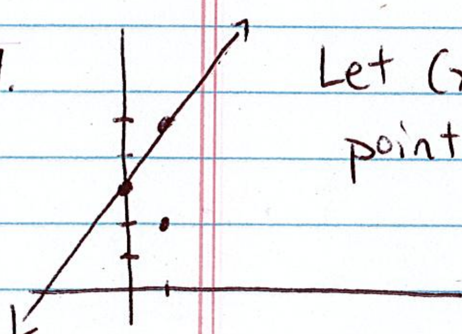
$\cos\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dy}{dt}$

$\cos\theta = \frac{8}{10} = \frac{4}{5}$

$\frac{4}{5}\frac{d\theta}{dt} = \frac{1}{10}(-2)$

$\frac{d\theta}{dt} = \frac{-1}{4}$ radian/sec

7.



Let (x, y) be any point on $y = 2x+3$

$d = \sqrt{(x-1)^2 + (y-2)^2}$

$\cancel{d^2 = (x-1)^2 + (y-2)^2}$

$\cancel{d' = \frac{1}{2}\left[(x-1)^2 + (y-2)^2\right]^{-1/2}}$

$\cancel{\left(2(x-1) + 2(y-2)\frac{dy}{dx}\right)}$

$d = \sqrt{(x-1)^2 + (2x+3-2)^2}$

$= \sqrt{(x-1)^2 + (2x+1)^2}$

min when $x = -0.2$, so $y = 2.6$

8. graph $g'(x)$ & look for the # of times $g'(x)$ crosses the x-axis, between $0 < x < 10$. $\boxed{11}$
↳ negative to positive

9. So $h(x)$ is always negative

$$g' \quad \underset{-3}{+ \,|\, -} \,\underset{3}{|}\, + \qquad \text{rel max at } x = -3$$

10. a) Relative minimums occur when f' changes from negative to positive, so at $x = -1$.
   b) Relative maximums occur when f' changes from positive to negative, so at $x = -5$.
   c) $f''(x) < 0$ means $f'(x)$ has negative slope, so $(-7, -3) \cup (2, 3) \cup (3, 5)$

11. $h'(x) = \dfrac{x^2 - 2}{x}$

   a) $h'(x) = 0$ when $x = \pm\sqrt{2}$

   local min at $x = -\sqrt{2}, \sqrt{2}$

   local max at $x =$ never

$$h' \quad - \underset{-\sqrt{2}}{|} + \underset{0}{|} - \underset{\sqrt{2}}{|} +$$
$(x \neq 0)$

   b) $h''(x) = \dfrac{x(2x) - (x^2-2)(1)}{x^2} = \dfrac{x^2+2}{x^2}$ (this is always positive!!)

   so concave up $(-\infty, 0) \cup (0, \infty)$
   (basically everywhere)

   c) $m = h'(4) = 3.5 \qquad y + 3 = \frac{7}{2}(x - 4)$

   d) Since $h''(4) > 0$, graph is concave up, so tangent line sits below the curve = underapproximation.

12. a) $2x + 4y\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-2x}{4y} = \frac{-x}{2y}$$

b) $\left.\dfrac{dy}{dx}\right|_{(1,2)} = \dfrac{-1}{4}$

$$y-2 = \frac{-1}{4}(x-1)$$

$$y-2 = \frac{-1}{4}(1.1-1)$$

$$y-2 = \frac{-1}{40}$$

$$y = 1\tfrac{39}{40}$$

b) $\dfrac{d^2y}{dx^2} = \dfrac{2y(-1) - (-x)(2\frac{dy}{dx})}{4y^2}$

$$\frac{d^2y}{dx^2} = \frac{-2y + 2x\left(\frac{-x}{2y}\right)}{4y^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,2)} = \frac{-4 + 2\left(\frac{-1}{4}\right)}{16} = \frac{-9}{32}$$

$\rightarrow$ which means curve is concave down, so <u>over</u> approximation

c) horizontal tangents when $x=0$.
$(0, \sqrt{\frac{9}{2}})$ & $(0, -\sqrt{\frac{9}{2}})$

e) vertical tangents when $y=0$.
$(3,0)$ & $(-3,0)$

13. a) $\lim_{x \to 0} \frac{e^x - 1}{-\sin x} = \lim_{x \to 0} \frac{e^x}{-\cos x} = \frac{1}{-1} = -1$

b) $\lim_{x \to a} \frac{1}{\frac{1}{x}} = \lim_{x \to a} x = a$

c) $\lim_{x \to -\infty} \frac{e^x}{x^3} = \lim_{x \to -\infty} \frac{e^x}{3x^2} = \lim_{x \to -\infty} \frac{e^x}{6x} = \lim_{x \to -\infty} \frac{e^x}{6} = 0$