NO CALCULATORS

1 **a** The common difference $d = -6$.

b $u_n = u_1 + (n-1)d$

$\therefore u_{20} = u_1 + 19d$

$= 51 + 19 \times -6$

$= 51 - 114$

$= -63$

c $S_n = \frac{n}{2}(u_1 + u_n)$

$\therefore S_{20} = \frac{20}{2}(51 - 63)$

$= 10 \times -12$

$= -120$

2 **a** The common ratio $r = 2$.

b $u_n = u_1 r^{n-1}$

$\therefore u_{20} = u_1 r^{19}$

$= \frac{1}{8} \times 2^{19}$

$= 2^{-3} \times 2^{19}$

$= 2^{16}$

c $S_n = \frac{u_1(r^n - 1)}{r-1}$

$\therefore S_{10} = \frac{\frac{1}{8}(2^{10} - 1)}{2-1}$

$= \frac{1023}{8}$

3 **a** $u_1 = 27$ and $u_4 = 8$

$\therefore u_1 r^3 = 8$

$\therefore r^3 = \frac{8}{27}$

$\therefore r = \frac{2}{3}$

b $u_n = u_1 r^{n-1}$

$\therefore u_6 = 27 \times \left(\frac{2}{3}\right)^5$

$= \frac{3^3 \times 2^5}{3^5}$

$= \frac{32}{9}$

c $S = \sum_{k=1}^{\infty} u_n = \sum_{k=1}^{\infty} 27 \times \left(\frac{2}{3}\right)^{k-1}$

d Since $|r| < 1$, the series converges.

$S = \frac{u_1}{1-r}$

$= \frac{27}{1 - \frac{2}{3}}$

$= 81$

4 **a** $u_7 = 1$ and $u_{15} = -23$

$\therefore u_1 + 6d = 1$ and $u_1 + 14d = -23$

$\therefore (u_1 + 14d) - (u_1 + 6d) = -23 - 1$

$\therefore 8d = -24$

$\therefore d = -3$

and $u_1 = 1 - 6d$

$= 19$

b $u_n = u_1 + (n-1)d$

$\therefore u_{27} = u_1 + 26d$

$= 19 + 26(-3)$

$= -59$

c $S_n = \frac{n}{2}(u_1 + u_n)$

$\therefore S_{27} = \frac{27}{2}(u_1 + u_{27})$

$= \frac{27}{2}(19 - 59)$

$= -540$

5 **a** $u_1 = 18$ and $d = -3$.

If the series has n terms, then

$S_n = -210$

$\therefore \frac{n}{2}(2u_1 + (n-1)d) = -210$

$\therefore \frac{n}{2}(2 \times 18 + (n-1) \times (-3)) = -210$

$\therefore \frac{n}{2}(36 - 3n + 3) = -210$

$\therefore \frac{n}{2}(39 - 3n) = -210$

b From **a**, $\frac{3}{2}n(13 - n) = -210$

$\therefore n(13 - n) = -140$

$\therefore n^2 - 13n - 140 = 0$

$\therefore (n - 20)(n + 7) = 0$

$\therefore n = 20$ {since $n > 0$}

6 **a** $\frac{x}{x^{-\frac{1}{2}}} = x^{1-(-\frac{1}{2})} = x^{\frac{3}{2}}$

$\frac{x^{\frac{5}{2}}}{x} = x^{\frac{5}{2}-1} = x^{\frac{3}{2}}$

$\therefore$ the common ratio $r = x^{\frac{3}{2}}$.

b $u_n = u_1 r^{n-1}$

$\therefore u_{10} = u_1 r^9$

$= x^{-\frac{1}{2}} \left(x^{\frac{3}{2}}\right)^9$

$= x^{-\frac{1}{2} + \frac{27}{2}}$

$= x^{13}$

c The corresponding infinite geometric series will converge provided $|r| < 1$, $r \neq 0$.

Now $r = x^{\frac{3}{2}}$, so r is only defined for $x > 0$, and $|r| = 1$ when $x = 1$.

$\therefore$ the series will converge for $0 < x < 1$.

7 **a** There are 7 terms in the series.

b $u_n = 3 \times 2^{n-1}$

$\therefore$ the first term $u_1 = 3 \times 2^0 = 3$
and the common ratio $r = 2$.

c $S_n = \frac{u_1(r^n - 1)}{r - 1}$

$\therefore S_7 = \frac{3(2^7 - 1)}{2 - 1}$

$= 3 \times 127$

$= 381$

8 **a** **i** $\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^k$ has general term

$u_n = 2\left(\frac{2}{3}\right)^n$

$= \frac{4}{3}\left(\frac{2}{3}\right)^{n-1}$

$\therefore$ the first term $u_1 = \frac{4}{3}$
and the common ratio $r = \frac{2}{3}$.

ii Since $|r| < 1$, the series converges.

$S = \frac{u_1}{1 - r}$

$= \frac{\frac{4}{3}}{1 - \frac{2}{3}}$

$= 4$

b **i** $\sum_{k=1}^{n}(k - 4)$ has general term $u_n = n - 4$

$\therefore$ the first term $u_1 = -3$
and the common difference $d = 1$.

ii $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$= \frac{n}{2}(2 \times -3 + (n-1)1)$

$= \frac{n}{2}(-6 + n - 1)$

$= \frac{n(n-7)}{2}$

c $\frac{n(n-7)}{2} = 4$

$\therefore n^2 - 7n - 8 = 0$

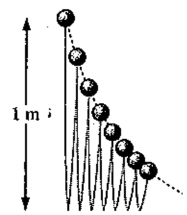
$\therefore (n - 8)(n + 1) = 0$

$\therefore n = 8$ {since $n > 0$}

9 **a** The series has first term $u_1 = 1$ and common ratio $r = 0.6$.
Since $|r| < 1$, the series converges.

$$S = \frac{u_1}{1-r} = \frac{1}{1-0.6} = \frac{5}{2}$$

b



The total distance travelled
$= 1 + 1 \times 0.6 \times 2$
$\quad + 1 \times (0.6)^2 \times 2$
$\quad + 1 \times (0.6)^3 \times 2 + \ldots.$
$= 1 + 0.6 \times 2(1 + 0.6 + (0.6)^2 + \ldots.)$
$= 1 + \frac{\cancel{6}^{3}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{2}_{1}}$ {using **a**}
$= 4$ metres

CALCULATORS

1 **a** $u_n = u_1 + (n-1)d$
$\quad = 100 + (n-1)30$

b We want to find n where $100 + 30(n-1) > 1200$
$\therefore \; 30(n-1) > 1100$
$\therefore \; n - 1 > 36.667$
$\therefore \; n > 37.667$
$\therefore$ the first term is u_{38} which is 1210.

c The sum of the first k terms is
$S_k = \frac{k}{2}(2u_1 + (k-1)d) = 19\,140$
$\therefore \; \frac{k}{2}(2 \times 100 + (k-1)30) = 19\,140$
$\therefore \; 15k^2 + 85k - 19\,140 = 0$
$\therefore \; k = 33$ or $-\frac{116}{3}$ {technology}
$\therefore \; k = 33$ $\{k \in \mathbb{Z}^+\}$

2 **a** $u_5 = u_1 r^4 = 18$ and $u_8 = u_1 r^7 = 486$
$\therefore \; \frac{u_1 r^7}{u_1 r^4} = \frac{486}{18}$
$\therefore \; r^3 = 27$ and so $r = 3$
Since $u_1 r^4 = 18$, $u_1 \times 81 = 18$
$\therefore \; u_1 = \frac{2}{9}$

b $u_n = u_1 r^{n-1}$
$\therefore \; u_{12} = u_1 r^{11}$
$\quad = \frac{2}{9} \times 3^{11}$
$\quad = 39\,366$

c $S_n = \frac{u_1(r^n - 1)}{r-1}$
$\therefore \; S_{10} = \frac{u_1(r^{10} - 1)}{r-1}$
$\quad = \frac{\frac{2}{9}(3^{10} - 1)}{3-1}$
$\quad \approx 6560$

3 **a** Her annual amount is geometric with $u_1 = 800$, $r = 107\% = 1.07$, and $n = 9$.
$u_9 = u_1 \times r^8 = 800 \times (1.07)^8$
$\quad \approx 1374.55$ euro

b We need to solve $800 \times (1.07)^n = 4000$
$\therefore \; (1.07)^n = 5$
$\therefore \; \log(1.07)^n = \log 5$
$\therefore \; n \log(1.07) = \log 5$
$\therefore \; n = \frac{\log 5}{\log(1.07)}$
$\therefore \; n \approx 23.79$
So, it will take 24 years.

4 **a** $u_1 = 10$
$u_2 = 10 \times 110\% = 10 \times 1.1$
$u_3 = u_2 \times 1.1 = 10 \times (1.1)^2$
$\vdots$
$u_7 = 10 \times (1.1)^6 = 17.715\,61$
So, Ying ran 17.7 km on day 7.

b Total distance ran $= u_1 + u_2 + u_3 + \ldots. + u_7$
which is geometric with $u_1 = 10$, $r = 1.1$, $n = 7$
$\therefore \; S_7 = \frac{10\left((1.1)^7 - 1\right)}{1.1 - 1}$
$\quad = \frac{10\left((1.1)^7 - 1\right)}{0.1}$
$\quad \approx 94.8717$
So, Ying ran a total of 94.9 km.

5 **a** $10 + 14 + 18 + \ldots. + 138$ is arithmetic with $u_1 = 10$, $d = 4$
Now $u_1 + (n-1)d = 138$
$\therefore \; 10 + 4(n-1) = 138$
$\therefore \; 4(n-1) = 128$
$\therefore \; n - 1 = 32$
$\therefore \; n = 33$

So, the sum is
$\frac{n}{2}(u_1 + u_{33})$
$= \frac{33}{2}(10 + 138)$
$= \frac{33}{2}(148)$
$= 2442$

b $6 - 12 + 24 - 48 + 96 - \ldots. + 1536$ is geometric with $u_1 = 6$, $r = -2$
Now $u_1 r^{n-1} = 1536$
$\therefore \; 6 \times (-2)^{n-1} = 1536$
$\therefore \; (-2)^{n-1} = 256$
$\therefore \; (-2)^{n-1} = (-2)^8$
$\therefore \; n - 1 = 8$
$\therefore \; n = 9$

So, the sum is
$\frac{u_1(1 - r^n)}{1-r}$
$= \frac{6(1 - (-2)^9)}{1 - (-2)}$
$= \frac{6}{3}\left(1 - (-2)^9\right)$
$= 2 \times 513$
$= 1026$

6 **a** $\frac{u_1}{1-r} = 1.5$
and $u_1 = 1$
$\therefore \; 1 - r = \frac{1}{1.5}$
$\therefore \; 1 - r = \frac{2}{3}$
$\therefore \; r = \frac{1}{3}$

b $S_n = \frac{u_1(1 - r^n)}{1-r}$
$\therefore \; S_7 = \frac{u_1(1 - r^7)}{1-r}$
$\quad = \frac{1\left(1 - (\frac{1}{3})^7\right)}{1 - \frac{1}{3}}$
$\quad = \frac{3}{2}\left(1 - \frac{1}{2187}\right)$
$\quad = \frac{1093}{729}$

7 **a** $\frac{u_{n+1}}{u_n} = \frac{12\left(\frac{2}{3}\right)^n}{12\left(\frac{2}{3}\right)^{n-1}}$
$\quad = \frac{2}{3}$ for all $n \in \mathbb{Z}^+$
Thus, the sequence is geometric with $r = \frac{2}{3}$.

b $u_5 = 12\left(\frac{2}{3}\right)^4$
$\quad = 12\left(\frac{16}{81}\right)$
$\quad = \frac{64}{27}$

c **i** $\sum_{n=1}^{\infty} u_n = \frac{u_1}{1-r}$
$\quad = \frac{12\left(\frac{2}{3}\right)^0}{1 - \frac{2}{3}}$
$\quad = \frac{12}{\frac{1}{3}}$
$\quad = 36$

ii $\sum_{n=1}^{20} u_n = S_{20}$
$S_n = \frac{u_1(1 - r^n)}{1-r}$
$\therefore \; S_{20} = \frac{12\left(1 - (\frac{2}{3})^{20}\right)}{1 - \frac{2}{3}}$
$\quad \approx 35.9892$

8 Time period = 33 months = 11 quarters

Interest rate = 8% p.a. = 2% per quarter

$\therefore \quad r = 1.02$

$\therefore$ the amount after 11 quarters is

$$u_{12} = u_1 \times r^{11}$$
$$= 3500 \times 1.02^{11}$$
$$\approx 4351.8101$$

So, the maturing value is £4351.81.

9 **a** The series is geometric with first term $u_1 = 12$, and common ratio $r = (x-2)$.

The series converges provided $|r| < 1$

$\therefore \quad 1 < x < 3$

b Since $1 < \sqrt{5} < 3$, the series converges.

$$S = \frac{u_1}{1-r}$$
$$= \frac{12}{1-(\sqrt{5}-2)}$$
$$\approx 15.7$$

10 **a** Pierre added $\$10 \times 8 = \$80$

Francesca added

$\$(0.50 + 1 + 1.50 + 2 + 2.5 + \dots + 4) = \$18$

b $u_{52} = u_1 + 51d$

$= 0.50 + 51 \times 0.50$

$= 26$ So, she added \$26.

c Pierre had $\$10 \times 52 + \$100 = \$620$

Francesca had $(0.50 + 1 + 1.50 + \dots + 26) + 100$

$= \frac{52}{2}(0.5 + 26) + 100$

$= 26 \times 26.5 + 100$

$= \$789$

11 **a** Hayley: $u_5 = u_1 + 4d = 60 + 4 \times 20$

$= 140$ km in week 5

Patrick: $u_5 = u_1 r^4 = 60 \times (1.2)^4$

≈ 124 km in week 5

b For Hayley $u_8 = 60 + 7 \times 20 = 200$

and $u_9 = 60 + 8 \times 20 = 220$

For Patrick $u_7 = 60 \times (1.2)^6 \approx 179$

and $u_8 = 60 \times (1.2)^7 \approx 215$

So, Patrick was the first to cycle 210 km in a week.

c Hayley: $S_{12} = \frac{12}{2}(2 \times 60 + 11 \times 20)$

$= 6 \times (120 + 220)$

$= 2040$ km

Patrick: $S_{12} = \dfrac{60\left(1.2^{12} - 1\right)}{1.2 - 1} \approx 2370$ km

12 **a** 2002: $u_1 = 2000$

2003: $u_2 = 2000 + 2000 \times 1.0825$

2004: $u_3 = 2000 + [2000 + 2000 \times 1.0825] \times 1.0825$

$= 2000 + 2000r + 2000r^2$ where $r = 1.0825$

$= 2000(1 + r + r^2)$

2009: Total amount $= 2000(1 + r + r^2 + r^3 + \dots + r^7)$

$$= 2000\left(\frac{r^8 - 1}{r - 1}\right)$$
$$= \frac{2000\left(1.0825^8 - 1\right)}{0.0825}$$
$$\approx 21\,466.32 \text{ rupees}$$

b Each 2000 rupee investment earns

$2000 \times 0.09 = 180$ rupees simple interest per year.

$\therefore$ total amount in 2009

$= 8 \times 2000 + 1 \times 180 + 2 \times 180 + \dots + 7 \times 180$

(8 deposits; interest from 2008 investment; interest from 2007 investment; interest from 2002 investment)

$= 16\,000 + \frac{7}{2}(180 + 1260)$

$= 21\,040$ rupees

So, Kapil will be 426.32 rupees better off with the compound interest option.

13 **a** The interest rate = 7.2% per annum

= 0.6% per month

$\therefore$ the value of the investment is multiplied by $r = 1.006$ each month.

b The value of the investment after n months is

$$u_{n+1} = u_1 \times r^n$$
$$= 500 \times 1.006^n$$

The value reaches €1000 when

$500 \times 1.006^n = 1000$

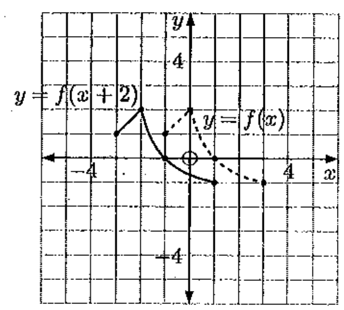
$\therefore \quad 1.006^n = 2$

$\therefore \quad n = \log_{1.006} 2$

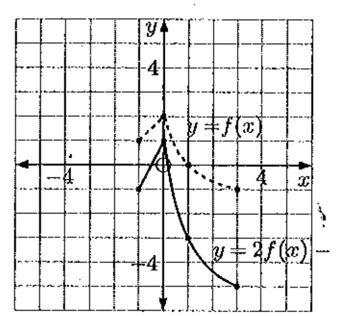
≈ 115.9 months

$\therefore$ it will take 116 months for Paige's investment to be worth €1000.

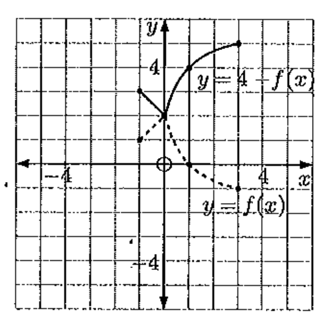
9 **a**



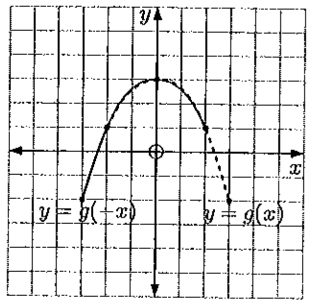
b



c



10 **a**



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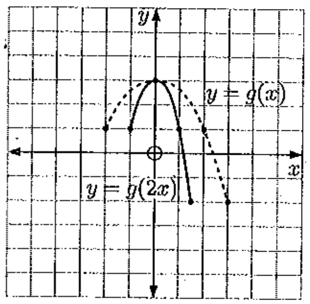


11 **a** Domain $= \{x \mid x > 2\}$, Range $= \{y \mid y \in \mathbb{R}\}$

b $y = g(x)$ has the vertical asymptote $x = 2$.

c $y = g(2x)$ is a horizontal stretch of g with scale factor $\frac{1}{2}$.

$\therefore\ h(x) = g(2x)$
$= 4 - \ln(2x - 2)$

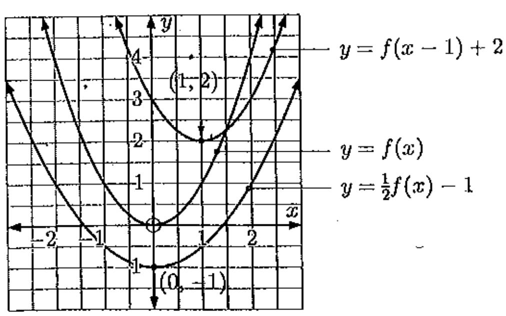
or $h : x \mapsto 4 - \ln(2x - 2)$

d $y = h(x)$ is defined when $2x - 2 > 0$

$\therefore\ x > 1$

$\therefore\ y = h(x)$ has the vertical asymptote $x = 1$.

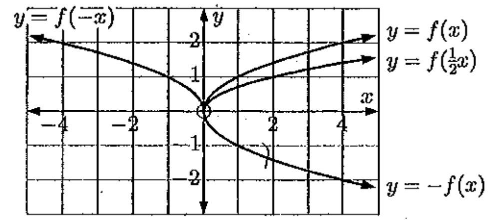
24 **a**



b $y = f(x - 1) + 2$ is a translation of $y = f(x)$ through $\binom{1}{2}$.

$y = \frac{1}{2}f(x) - 1$ is a vertical stretch of $y = f(x)$ with scale factor $\frac{1}{2}$, followed by a vertical translation of 1 unit downwards.

25 **a**



b $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis.
$y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis.
$y = f\left(\frac{1}{2}x\right)$ is a horizontal stretch of $y = f(x)$ with scale factor 2.