May 2015 Timezone 2
IB Maths Standard Exam
Worked Solutions

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Paper 1

Problem 1

Solution 1 (a)

The probability that the first marble drawn is red is the number of red marbles divided by the total number of marbles: $\boxed{\frac{3}{8}}$.

Solution 1 (b)

The solution from part (a) is the probability of drawing a red marble on the first pick. Therefore, that value goes in the empty slot for the "Red" branch of the "First Marble" event in the tree diagram. Since the probabilities from all the branches of each tree level need to add up to one (1), we can also validate that this answer makes sense because $\frac{3}{8} + \frac{5}{8} = 1$.

For the second marble pick, since we are told that the two selections are done *without replacement*, we know there are only seven marbles left. If a red marble was picked on the first turn, then we have 2 red marbles and five blue left. If a blue marble was picked on the first turn, then we have 3 red marbles and four blue left. This information lets us fill in the branches of the second level of the tree diagram as shown in Figure 1.

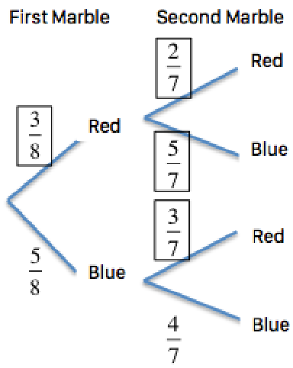


Figure 1: Completed Tree Diagram

Solution 1 (c)

To find the probability that both marbles are blue, we multiply the probability of drawing a marble on the first pick $\frac{5}{8}$ by the probability of drawing a blue marble on the second turn (given a blue was picked on the first turn), which is $\frac{4}{7}$. Therefore:

$$P(\text{Both Blue}) = P(\text{Blue on First}) \times P(\text{Blue on Second} \mid \text{Blue on First}) = \frac{5}{8} \times \frac{4}{7} = \frac{20}{56} = \boxed{\frac{5}{14}}$$

Problem 2

Solution 2(a)(i)

The graph of sin shown completes one cycle from zero (0) to π before repeating again, therefore $\boxed{\text{the period is } \pi}$.

Solution 2(a)(ii)

The graph of sin shown goes up to a maximum of 3 and down to a minimum of -3. The amplitude is half of the distance between the maximum and the minimum value, so $\dfrac{3-(-3)}{2} = \dfrac{6}{2} = 3$. So the $\boxed{\text{amplitude is } 3}$

Solution 2(b)(i)

The value of a in the expression $f(x) = a\sin(bx)$ *is* the amplitude, so it is just the value you computed in 2a(ii) above. $\boxed{a = 3}$.

Solution 2(b)(ii)

Since we observed in 2a(i) that the period is π, the value of b is given by the formula:

$$\frac{2\pi}{b} = \pi$$
$$\frac{2}{b} = 1$$
$$b = 2$$

So, the value of $\boxed{b = 2}$

Problem 3

Solution 3(a)

To find the median length of the fish, we find the midpoint on the vertical "Cumulative Frequency" axis of the graph and use the curve to determine the corresponding value on the "Length" axis. The midpoint of the frequency is $\frac{160}{2} = 80$. Drawing a horizontal line from 80 across to meet the curve, and then drawing a vertical line down from this intersection point to meet the "Length" axis, we find $\boxed{\text{the median is } 3\,\text{cm}}$.

Solution 3(b)(i)

The value of p is the cumulative frequency of fish having lengths of 2 cm or smaller. Using the cumulative frequency graph given, drawing a vertical line up from the value of 2 on the "Length" axis until it meets the curve, then drawing a horizontal line across to meet the "Cumulative Frequency" axis, we find that the value of $\boxed{\text{p is } 30\,\text{cm}}$.

Solution 3(b)(ii)

The value of q is the cumulative frequency of fish having lengths between 3 and 4.5 cm. This will be the difference between the cumulative frequency of fish having lengths 3 or smaller, and that of fish having length 4.5 or smaller. The graph shows that the cumulative frequency of fish having length 3 or smaller is 80. It also shows us that the cumulative frequency of fish having length 4.5 or smaller is 140. So the cumulative frequency of fish with lengths between 3 and 4.5 will be $140 - 80 = 60$. Therefore, $\boxed{q \text{ is } 60\,\text{cm}}$

To double-check our work, we can add up the values in the table now that we have computed p and q to make sure they all add up to 160. We verify this is true by adding $p + 50 + q + 20 = 30 + 50 + 60 + 20 = 160$.

Problem 4

Solution 4(a)

The function $g(x) = \frac{\ln x}{x}$ is a quotient of two functions involving x, so to find the first derivative $g'(x)$, we let $u(x) = \ln x$ and $v(x) = x$ and rewrite $g(x) = \frac{u(x)}{v(x)}$, then use the quotient rule.

$$u(x) = \ln x \qquad\qquad v(x) = x$$

$$\frac{d\,u}{dx} = \frac{1}{x} \qquad\qquad \frac{d\,v}{dx} = 1$$

By the quotient rule, we have:

$$g'(x) = \frac{v(x) \cdot \frac{d\,u}{dx} - u(x) \cdot \frac{d\,v}{dx}}{(v(x))^2} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \boxed{\frac{1 - \ln x}{x^2}}$$

Solution 4(b)

To compute the anti-derivative of $g(x)$ we let $u(x) = \ln x$. Computing the derivative of this substitution, we get:

$$\begin{aligned} u(x) &= \ln x \\ du &= \frac{1}{x}\, dx \end{aligned}$$

So we can rewrite the integral:

$$\int \frac{\ln x}{x}\, dx = \int \ln x \cdot \frac{1}{x}\, dx = \int u\, du$$

Computing the anti-derivative of u gives us $\dfrac{u^2}{2} + C$. Substituting our expression in x back in for u, this gives the final answer:

$$\int \frac{\ln x}{x}\, dx = \boxed{\frac{(\ln x)^2}{2} + C}$$

Problem 5

Solution 5(a)

Since $f(x) = \mathrm{e}^{-2x}$, using the chain rule we have:

$$\begin{aligned} f'(x) &= \mathrm{e}^{-2x} \cdot (-2) = \boxed{-2\mathrm{e}^{-2x}} \\ f''(x) &= -2\mathrm{e}^{-2x} \cdot (-2) = \boxed{4\mathrm{e}^{-2x}} \\ f^{(3)}(x) &= 4\mathrm{e}^{-2x} \cdot (-2) = \boxed{-8\mathrm{e}^{-2x}} \end{aligned}$$

Solution 5(b)

We can observe above that each of the first three derivatives has the same factor e^{-2x} so that must appear in the n^{th} derivative as well. We observe that the first derivative has a coefficient of $(-2)^1 = 2$, the second derivative has a coefficient of $(-2)^2 = 4$, and the third derivative has the coefficient of $(-8)2^3 = -8$. So it appears that the n^{th} derivative will have the coefficient of $(-2)^n$. Therefore, we have:

$$f^{(n)}(x) = \boxed{(-2)^n \mathrm{e}^{-2x}}$$

Problem 6

We have $f(x) = ax^3 + bx$ and they tell us that $f'(0) = 3$. They also tell us that $f^{-1}(7) = 1$. If the function f^{-1} maps the number 7 to the number 1, then its inverse function f must "undo" this mapping. In other words, it must be the case that the function $f(x)$ maps the number 1 to the number 7, that is $f(1) = 7$. Using this fact, we have the equation:

$$\begin{aligned} f(1) &= 7 \\ a(1)^3 + b(1) &= 7 \\ a + b &= 7 \end{aligned}$$

We also know that $f'(0) = 3$, so we first take the derivative of $f(x)$ to get $f'(x) = 3ax^2 + b$ and then substitute in zero for x to get this equation:

$$\begin{aligned} f'(0) &= 3 \\ 3a(0)^2 + b &= 3 \\ b &= 3 \end{aligned}$$

Therefore, $\boxed{b = 3}$ and substituting this back into the first equation $a + b = 7$ gives $a + 3 = 7$, so $\boxed{a = 4}$.

Problem 7

If a game is fair, we know that the expected value of the player's winnings should be zero. If Rose draws a white chip (0.6 probability) then the problem says that she "gets no money", but since she has to pay \$10 to play, this means that in this case her winnings are -\$10.00. That is, since the loses \$10, it's the same as saying that she wins -\$10.00. If she draws a black chip (0.4 probability) then the problem says that she gets k dollars. However, again since she had to pay \$10.00 to play, her winnings in this case are $k - 10$ dollars.

The expected value of her winnings is just the sum of the winnings in each case, multiplied by the probability of that case's occurring. Since the game is fair, this expected value must equal zero. That gives this equation:

$$\begin{aligned} (-10)(0.6) + (k - 10)(0.4) &= 0 \\ -6 + 0.4k - 4 &= 0 \\ 0.4k &= 10 \\ k &= \frac{10}{0.4} = 25 \end{aligned}$$

Therefore, Rose must get $\boxed{\text{k = 25 dollars}}$ for picking a black chip.

Problem 8

Solution 8(a)(i)

We're told that $f(x) = a(x + 3)(x - 1)$, and that the graph has x-intercepts at $(p, 0)$ and $(q, 0)$. Since the function is already in x-intercept form, we can read off the values that would make $f(x)$ equal to zero. Those are the values that would make $x + 3 = 0$ or $x - 1 = 0$, so -3 or 1. Since we can see by the graph that p is to the left of the y-axis, that means its value must be the negative one, so we can infer that $\boxed{p = -3 \text{ and } q = 1}$.

Solution 8(a)(ii)

To find a we need to use some point on the graph other than the two zeros, since infinitely many quadratic equations can pass through those two zeros. They give us the y-intercept in the problem, telling us that $f(0) = 12$. We can use this information to find the value of a now that we already know p and q. Just fill in all the information we already know, and a is the only value left to solve for.

$$\begin{aligned} f(0) &= 12 \\ a(0 + 3)(0 - 1) &= 12 \\ a \cdot 3 \cdot (-1) &= 12 \\ a &= \frac{12}{-3} = -4 \end{aligned}$$

Therefore, $\boxed{a = -4}$

Solution 8(b)

In a quadratic equation with two distinct roots, the axis of symmetry is the vertical line with the x coordinate value that is half way between the two zeros. Taking the average of the x coordinate values of the two zeros p and q above, gives the midpoint we need:

$$\frac{-3+1}{2} = \frac{-2}{2} = -1$$

So the equation of the axis of symmetry is the vertical line $\boxed{x = -1}$.

Solution 8(c)

Since we can see from the graph that the parabola is concave down, the largest value of the function $f(x)$ occurs at the vertex. We computed the x coordinate of the vertex in the previous part, so the value of the function at this point is:

$$f(-1) = -4(-1+3)(-1-1) = -4(2)(-2) = \boxed{16}.$$

Solution 8(d)

The function can also be written in vertex form as $f(x) = a(x-h)^2 + k$, where h and k are the x and y coordinates of the vertex, respectively. We computed in part 8b that the x coordinate of the vertex is -1 and in part 8c we computed the y coordinate of the vertex is 16. So $h = -1$ and $k = 16$.

Problem 9

Solution 9(a)

We are told that $\vec{\boldsymbol{P}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\vec{\boldsymbol{Q}} = \begin{pmatrix} -11 \\ 8 \\ m \end{pmatrix}$. The vector $\overrightarrow{\boldsymbol{PQ}}$ from P to Q is $\vec{\boldsymbol{Q}} - \vec{\boldsymbol{P}}$ so:

$$\overrightarrow{\boldsymbol{PQ}} = \vec{\boldsymbol{Q}} - \vec{\boldsymbol{P}} = \begin{pmatrix} -11 \\ 8 \\ m \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 \\ 8 \\ m-2 \end{pmatrix}$$

Solution 9(b)

We are told that $\vec{\boldsymbol{a}} = \begin{pmatrix} 1 \\ 1 \\ n \end{pmatrix} \perp \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \vec{\boldsymbol{b}}$. When two vectors are perpendicular, their dot product equals zero, so we use this fact to solve for n:

$$\begin{aligned} \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} &= 0 \\ \begin{pmatrix} 1 \\ 1 \\ n \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} &= 0 \\ 1(-3) + 1(2) + n \cdot 1 &= 0 \\ -3 + 2 + n &= 0 \\ n &= 1 \end{aligned}$$

Solution 9(c)(i)

If $\overrightarrow{\boldsymbol{PQ}}$ is parallel to $\vec{\boldsymbol{b}}$ then $\overrightarrow{\boldsymbol{PQ}}$ must be a scalar multiple of $\vec{\boldsymbol{b}}$. That is, $\overrightarrow{\boldsymbol{PQ}} = k\vec{\boldsymbol{b}}$ for some scalar value of k. Using our answer to 9(a), this means that:

$$\overrightarrow{\boldsymbol{PQ}} = \begin{pmatrix} -12 \\ 8 \\ m-2 \end{pmatrix} = k\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3k \\ 2k \\ k \end{pmatrix}$$

Since equal vectors must have equal coordinates, we can conclude that:

$$\begin{aligned} -12 &= -3k \\ 8 &= 2k \\ m-2 &= k \end{aligned}$$

We can solve either of the first two equations above to conclude that $k = 4$, then substitute that into the third equation to get:

$$m - 2 = 4m \qquad\qquad = 6$$

Solution 9(c)(ii)

Using the fact from the previous part, we have:

$$\overrightarrow{\boldsymbol{PQ}} = \begin{pmatrix} -12 \\ 8 \\ m-2 \end{pmatrix} = k\vec{\boldsymbol{b}} = k\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3k \\ 2k \\ k \end{pmatrix}$$

These vectors are equal if and only if each of their components is equals, so the above gives the following set of equations:

$$\begin{aligned} -12 &= -3k \\ 8 &= 2k \\ m-2 &= k \end{aligned}$$

From either of the first two equations, we can infer that $k = 4$, so we can substitute that value for k into the third equation to get $m - 2 = 4$. Therefore, $m = 6$.

Solution 9(d)(i)

We're told that a particle moves along a "straight line through Q" so that it's position is given by $\vec{\boldsymbol{r}} = \vec{\boldsymbol{c}} + t\vec{\boldsymbol{a}}$. If the vector line goes through $\vec{\boldsymbol{Q}}$ then we can use $\vec{\boldsymbol{Q}}$ as the position vector of the vector line. Therefore, a possible vector $\vec{\boldsymbol{c}}$ for the vector equation is the vector $\vec{\boldsymbol{Q}} = \begin{pmatrix} -11 \\ 8 \\ 6 \end{pmatrix}$.

Solution 9(d)(ii)

The speed of the particle is given by the magnitude of the direction vector of the line along which the particle is moving. So the magnitude of $\vec{\boldsymbol{a}} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}\,\mathrm{m/s}$

Problem 10

Solution 10(a)

The function f is defined on the domain $-3 < x < 3$. The graph of $f'(x)$ has x-intercepts at $x = a$, $x = 0$, and $x = d$. There is a local maximum (of $f'(x)$) at $x = b$, and local minima (of $f'(x)$) at $x = a$ and $x = c$.

The problem asked to find all possible values of x where the graph of f is decreasing. We know that f is decreasing exactly wherever its first derivative $f'(x)$ is negative, since this first derivative gives us the slope of f at any value of x, and a negative slope at a point indicates that the function is decreasing at that point. Since we have been given a graph of f', we need only look for the intervals where f' is below the x-axis to identify all the places where f' has a negative value, and hence where the original function f is decreasing. We can see that $f\prime(x)$ is negative on the interval $0 < x < d$.

Solution 10(b)(i) and 10(b)(ii)

The graph of f will have a local minimum at a point where $f'(x) = 0$ and where the value of f' changes from negative to positive. On the graph, we see that at $\boxed{x = d}$ the graph of $f' = 0$ and that at this point the values of $f'(x)$ nearby change from being below the x-axis (negative) to being above the x-axis (positive). This is the only value of x in the domain where these criteria are met.

Solution 10(c)

We are told that the total area of the region enclosed by the graph of f' and the x-axis is 15, and that $f(a) = 3$ and $f(d) = -1$. We need to find the value of $f(0)$.

We can represent the total area under the graph from a to 0 by the integral:

$$\int_a^0 f'(x)\,dx$$

Similarly, we can represent the total area under the graph from 0 to d by the integral:

$$\int_0^d -f'(x)\,dx$$

We add a minus sign in front of the $f'(x)$ because we want the area to come out positive. We are effectively calculating the area between two curves where the curve "above" is the x-axis with equation $x = 0$ and the curve "below" is $f'(x)$. So the area between these two curves from 0 to d is the integral of the difference $0 - f'(x)$ or just $-f'(x)$.

This means that the total area under the curve is just the sum of these two expressions, and we are told that it equals 15.

$$\int_a^0 f'(x)\,dx + \int_0^d -f'(x)\,dx = 15$$

If we invert the boundaries of the second integral, we know that will cause the answer to be the opposite (negative) of what it was before, so we can rewrite this as:

$$\int_a^0 f'(x)\,dx + \int_d^0 f'(x)\,dx = 15$$

Recall the Fundamental Theorem of Calculus. It tells us that the area under the curve of the *derivative of* f from p to q is equal to $f(q) - f(p)$. That is:

$$\int_p^q f'(x)\,dx = f(q) - f(p)$$

Using this fact, we can simplify both integrals above:

$$\begin{aligned}
\int_a^0 f'(x)\,dx + \int_d^0 f'(x)\,dx &= 15 \\
(f(0) - f(a)) + (f(0) - f(d)) &= 15 \\
2f(0) - f(a) - f(d) &= 15
\end{aligned}$$

Then, we can substitute into the equation the values of $f(a)$ and $f(d)$ that we were given:

$$\begin{aligned}
2f(0) - f(a) - f(d) &= 15 \\
2f(0) - 3 - (-1) &= 15 \\
2f(0) - 2 &= 15 \\
2f(0) &= 17 \\
f(0) &= \frac{17}{2}
\end{aligned}$$

1 Paper 2

Problem 1

Solution 1(a)

The problem gives a triangle with two angles and one side. We can use the Law of Sines to determine the length of side AC.

$$\begin{aligned}
\frac{AC}{\sin 80^\circ} &= \frac{10}{\sin 35^\circ} \\
AC &= \frac{\sin 80^\circ \cdot 10}{\sin 35^\circ} \\
&= \frac{0.984808 \cdot 10}{0.573576} \\
&= 17.1696 \\
&\approx 17.2\,\text{cm} \quad \text{(3sf)}
\end{aligned}$$

Solution 1(b)

To find the area of the triangle we need two sides and the included angle. Since we just determined the length of AC, we can use side AC (=17.1696), side BC (=10), and the included angle (which is $180 - 35 - 80 = 65^\circ$). Using the formula for the area of the triangle from the formula book, and substituting what we know:

$$\begin{aligned}
Area &= \frac{1}{2}(AC)(BC)\sin 65^\circ \\
&= \frac{1}{2}(17.1696)(10) \cdot 0.906308 \\
&= 77.8047 \quad \text{cm}^2 \\
&\approx 77.8 \quad \text{cm}^2 \quad \text{(3sf)}
\end{aligned}$$

Problem 2

Solution 2(a)(i)

We're told $\vec{\boldsymbol{u}} = \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix}$ and $\vec{\boldsymbol{v}} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$. So we have:
$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = 6 \cdot 2 + 3 \cdot 2 + 6 \cdot 1 = 12 + 6 + 6 = 24$

Solution 2(a)(ii)

$|u| = \sqrt{6^2 + 3^2 + 6^2} = \sqrt{36 + 9 + 36} = \sqrt{81} = 9$

Solution 2(a)(iii)

$|v| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$

Solution 2(b)

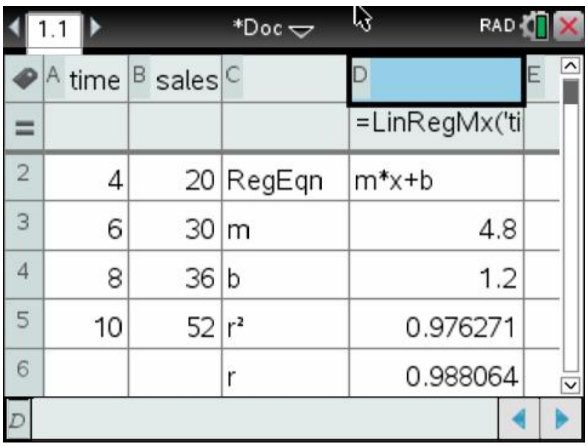
To find the angle between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ we use the formula for the cosine of the angle between two vectors, fill in the values we already computed above, then take the inverse cosine of that:

$$\begin{aligned}
\cos\theta &= \frac{\vec{u} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{u}}|\,|\vec{\boldsymbol{v}}|} \\
&= \frac{24}{9 \cdot 3} \\
&= \frac{24}{27} \\
\theta &= \cos^{-1}\left(\frac{24}{27}\right) = \cos^{-1}\left(\frac{8}{9}\right) \\
&= 0.475882 \quad \text{radians} \\
&\approx 0.476 \quad radians
\end{aligned}$$

Problem 3

Solution 3(a)(i)

Using a spreadsheet and the regression line calculation in the TI-NSpire CX, produces coefficients as shown Figure 2.



	A time	B sales	C	D	E
=				=LinRegMx('ti	
2	4	20	RegEqn	m*x+b	
3	6	30	m	4.8	
4	8	36	b	1.2	
5	10	52	r²	0.976271	
6			r	0.988064	

Figure 2: Results from Computing Regression Line

Therefore, we have $a = 4.8$ and $b = 1.2$

Solution 3(a)(ii)

Reading off the regression line output, we have $r = 0.988$

Solution 3(b)

The regression line found above is $y = 4.8x + 1.2$, so when $x = 7$ we have $y = 4.8(7) + 1.2 = 34.8$ million dollars.

Problem 4

The third term in the expansion of $(x + k)^8$ is $63x^6$. The third term is the one with $r = 2$, so using the Binomial Theorem, we compute the complete third term:

$$\binom{8}{2} x^{8-2} k^2 = 28x^6 k^2 = (28k^2)x^6$$

We need to find the values of k that would make $28k^2 = 63$, so we have:

$$\begin{aligned} 28k^2 &= 63 \\ k^2 &= \frac{63}{28} = \frac{9}{4} \\ k &= \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2} \end{aligned}$$

Problem 5

Solution 5(a)

We're given the function $f(x) = \mathrm{e}^{x+1} + 2$ for the domain $-4 \le x \le 1$. Using the TI-NSpire CX the graph looks like Figure 3. Notice I used the trace function to compute and plot the points at the ends of the domain boundaries (-4,2.05), (1,9.39), as well as the y-intercept of (0,4.72).

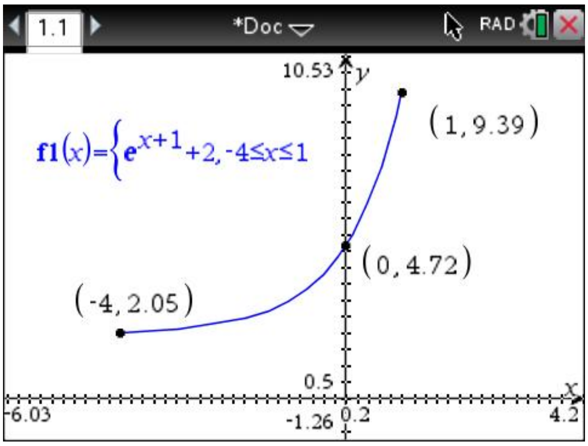


Figure 3: Graph of $f(x)$ with 3 Traced Points

Solution 5(b)

If the graph of f gets translated by the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ then that means that it is shifted 3 units in the x direction and -1 units in the y direction. In other words, it is shifted 3 units to the right and 1 unit down. In terms of function transformations, this means:

$$\begin{aligned} g(x) &= f(x-3) - 1 \\ g(x) &= \left(\mathrm{e}^{(x-3)+1} + 2\right) - 1 \\ g(x) &= \mathrm{e}^{x-2} + 1 \end{aligned}$$

Problem 6

The distances that Ramiro walks each minute form a sequence, with each subsequent term equal to 0.90 times the previous term: $u_{n+1} = u_n(0.90)$. Using this information we see that the sequence comprises values like: $u_1 = 80$, $u_2 = 80(0.90)$, $u_3 = 80(0.90)^2$, $u_4 = 80(0.90)^3$, etc. So the sequence of distances is a geometric sequence with $u_1 = 80$ and $r = 0.90$.

The simplest way to answer the question is compute S_{15} and show that after 15 minutes he won't yet have walked 660 metres. To do that, we compute:

$$\begin{aligned} S_{15} &= \frac{u_1\left(1-0.90^{15}\right)}{(1-0.90)} \\ &= \frac{80\left(1-0.90^{15}\right)}{(1-0.90)} \\ &= 635.287 \quad \text{metres} \end{aligned}$$

Therefore, after 15 minutes at 08:15 he will have only walked slightly more than 635 metres, so he won't yet be at school which was 660 metres from home.

Another way you might have thought to solve the problem is to use the formula for the partial sum of a geometric series to solve for how many minutes it will take Ramiro to walk 660m.

$$\begin{aligned} 660 &= \frac{u_1\left(1-r^n\right)}{(1-r)} \\ 660 &= \frac{80\left(1-0.90^n\right)}{(1-0.90)} \\ \frac{660}{80} &= \frac{1-0.90^n}{0.1} \\ \frac{660(0.1)}{80} &= 1-0.90^n \\ 0.90^n &= 1 - \frac{660(0.1)}{80} \\ 0.90^n &= \frac{7}{40} \\ \log_{0.90}\left(0.90^n\right) &= \log_{0.90}\frac{7}{40} \\ n &= \log_{0.90}\frac{7}{40} = \frac{\ln\frac{7}{40}}{\ln 0.90} = 16.5429 \end{aligned}$$

So, it will take Ramiro slightly more than 16.5 minutes to walk 660 metres, therefore after only 15 minutes at 08:15 he will not have arrived yet.

Problem 7

We're given two functions $f(x) = kx^2 + kx$ and $g(x) = x - 0.8$ and told they intersect in two distinct points. At the outset we need to recognize that the eventual possible value(s) for k that we'll find must be non-zero, otherwise if $k = 0$ then $f(x) = 0 \cdot x^2 + 0 \cdot x = 0$ which would make $f(x)$ just be the equation of the x-axis in which case $f(x)$ would intersect the line $g(x)$ only at the single point $(0.8, 0)$. So, we can rule out $k = 0$ as one of the possible values of k that cause $f(x)$ and $g(x)$ to intersect in two distinct places.

To find where the equations intersect, we set the equations equal and solve for x:

$$\begin{aligned} kx^2 + kx &= x - 0.8 \\ kx^2 + kx - x + 0.8 &= 0 \\ kx^2 + (k-1)x + 0.8 &= 0 \end{aligned}$$

If the two intersect in two distinct points, then there must be two distinct value of x that are roots for the above quadratic equation. We can use this fact, along with our knowledge of the discriminant, to solve for possible values of k.

We know that if there are two distinct roots in the equation above, that its discriminant must be greater than zero. So, we start by identifying the coefficients of the quadratic equation above: $a = k$, $b = k - 1$, and $c = 0.8$. The discriminant is the expression $b^2 - 4ac$, and this must be greater than zero:

$$\begin{aligned} (k-1)^2 - 4 \cdot k \cdot 0.8 &> 0 \\ k^2 - 2k + 1 - 3.2k &> 0 \\ k^2 - 5.2k + 1 &> 0 \end{aligned}$$

This is a new quadratic equation to solve to find the values of k that satisfy this condition. The easiest way to solve this is to graph it and find the intervals where the parabola is above the x-axis. The graph looks like Figure 4, and we've also used the "Analyze Graph" to find the two zeros at $k = 0.2$ and $k = 5$. We can see from the graph that the two intervals where the parabola is above the x-axis are $k < 0.2$ or $k > 5$ (remembering from our comment at the beginning of this problem that $k \neq 0$.

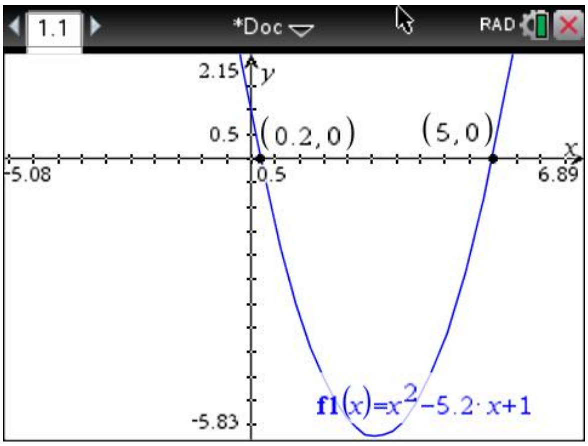


Figure 4: Graph of $k^2 - 5.2k + 1$ Showing Zeros

Another, slightly more laborious way we can solve the problem is using the quadratic equation:

$$\begin{aligned}
k &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{5.2 \pm \sqrt{(-5.2)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\
&= \frac{5.2 \pm \sqrt{27.04 - 4}}{2} \\
&= \frac{5.2 \pm \sqrt{23.04}}{2} \\
&= \frac{5.2 \pm 4.8}{2} \\
&= \frac{5.2 + 4.8}{2} \text{ or } \frac{5.2 - 4.8}{2} \\
&= 5 \,\text{or}\, 0.2
\end{aligned}$$

So this means that we can factor the quadratic equation in k above as follows, and that the values $k = 5$ and $k = 0.2$ are the values that would make the expression equal to zero.

$$\begin{aligned}
k^2 - 5.2k + 1 > 0 \\
(k - 5)(k - 0.2) > 0
\end{aligned}$$

We consider the number line with points $k = 5$ and $k = 0.2$ and try values in the three intervals determined by these two points to see whether the overall expression is greater than zero or not.

The thee intervals to try are $k < 0.2$, $0.2 < k < 5$, and $k > 5$. If we pick the simplest value to try in each interval, we can determine the sign of the result:

- In the interval $k < 0.2$, we try the value zero (0) and get $(0-5)(0-0.2)$ which is a negative times a negative. This is a positive number, so any value in the interval $k < 0.2$ yields an expression greater than zero.
- In the interval $0.2 < k < 5$, we try the value 1 and get $(1-5)(1-0.2)$ which is a negative times a positive. This yields a result that is less than zero.
- In the interval $k > 5$, we try the value 6 and get $(6-5)(6-0.2)$ which is a positive times a positive. This is a positive number.

Therefore, the two intervals in which we can pick values of k to make the discriminant greater than zero – thus ensuring that the original equation $kx^2 + (k-1)x + 0.8$ to has two distinct zeros – are $k < 0.2$ or $k > 5$ (and recalling the comment from the opening paragrah, $k \neq 0$).

Problem 8

Solution 8(a)

We're told that $f(x) = \frac{9}{x+2}$ and $g(x) = 3x^2$ for $x \geq 0$. To find the value of the intersection of the two curves we graph both of them and use the calculator's "Analyze Graph" feature to find the intersection of (1,3) shown in Figure 5. Therefore $p = 1$ and $q = 3$.

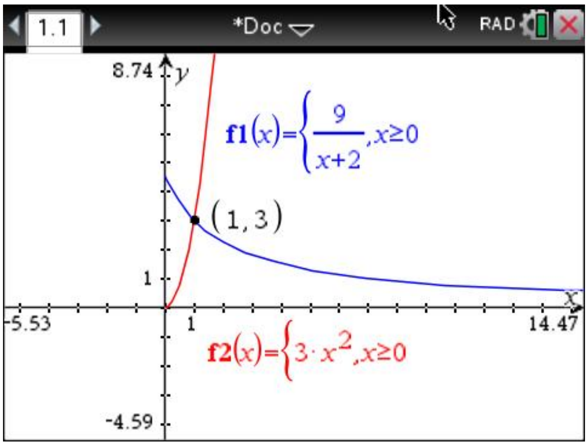


Figure 5: Graph of $k^2 - 5.2k + 1$ Showing Zeros

Solution 8(b)

Using the calculator to compute the derivative of a function at a point, we evaluate the numerical derivative of $\frac{9}{x+2}$ at the point $x = 1$ to get the answer -1 as shown in Figure 6.

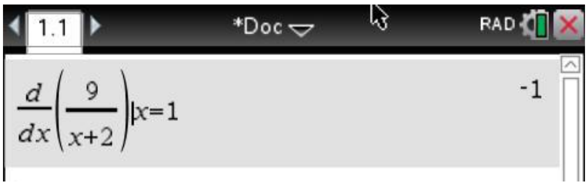


Figure 6: Numerical Derivative of $\frac{9}{x+2}$ at $x = 1$

Solution 8(c)(i)

If L is the normal to the graph of f at P, then its slope is the negative reciprocal of the slope of the tangent line to f at $P = (1, 3)$. The slope of the tangent line to f at P is given by the first derivative $f'(1)$, which we just computed above is -1. The negative reciprocal of this is positive 1. The line L goes through the point $(1, 3)$ and has slope 1, so using the point-slope form of a line, we can write the equation of L as shown below, then we can manipulate the equation to put it into the slope-intercept form that the problem is asking for.

$$\begin{aligned} y - 3 &= 1\,(x - 1) \\ y &= x - 1 + 3 \\ y &= x + 2 \end{aligned}$$

Solution 8(c)(ii)

We wrote the answer to the previous part in slope-intercept form, so we can read off the y-intercept of L is 2. In other words, the y-intercept is the point $(0, 2)$.

Solution 8(d)

Figure 7 shows the region R whose area we need to compute. This is the area between the curve L (above) and curve $g(x) = 3x^2$ below between the boundaries from from $x = 0$ to $x = 1$, which we can calculate using the integral:

$$\int_0^1 L - g(x)\,dx = \int_0^1 (x + 2) - 3x^2\,dx$$

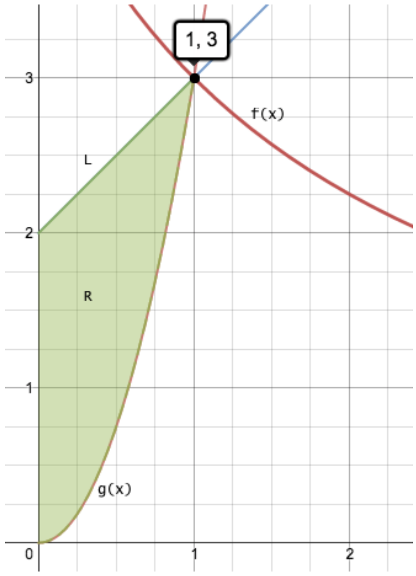


Figure 7: Region R Between L, g, and the y-axis

The integral would be simple enough to compute manually, but since this is a calculator test we can enter it directly into the calculator as shown in Figure 8 to get the answer of 1.5.

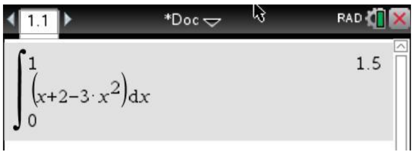


Figure 8: Integral to Compute Area of Region R

Problem 9

I don't recall having ever encountered the notation $N(50, \sigma^2)$ in any of the past papers I worked through with my students, but I went back to check in both their Pearson and Cambridge University Press textbooks and the notation of $X \sim N(\mu, \sigma^2)$ is mentioned. Every other past paper problem we worked always spelled out the normal distribution by saying "normally distributed with a mean μ and standard deviation σ", so that was what my students had been used to. Also, the fact that this problem uses a random variable L instead of the normal variable name X that students were used to seeing was another thing that threw some students a curveball. Remember that in the $N(\mu, \sigma^2)$ the second number is the *variance*, which is the square of the standard deviation σ.

So, the statement in the problem that $L \sim N(50, \sigma^2)$, means that the random variable L (for the *length* of the nails) is normally distributed with a mean of 50 mm and a variance of σ^2. This means that it has a standard deviation of the square root of this variance, which is σ.

Solution 9(a)

We're asked to find $P((50 - \sigma < L < 50 + 2\sigma)$. Again, this is notation that was unfamiliar to my students, but it's essentially asking "What is the probability that the length L of a randomly selected nail will be between one standard deviation below the mean and two standard deviations above the mean?"

Recall, as shown in Figure 9 that the probability in a normal distribution is symmetric about the mean, and that 68% of the area falls within one standard deviation of the mean, 95% falls within two standard deviations of the mean, and 99% falls within three standard deviations of the mean. Therefore the probability that L falls within one standard deviation below the mean and two standard deviations above the mean is $34\% + 34\% + 13.5\% = 81.5\%$ or as a decimal 0.815.

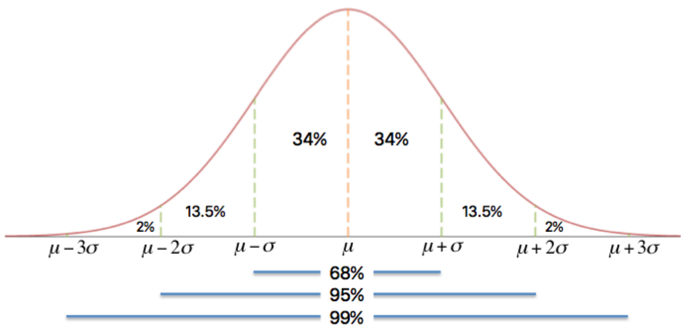


Figure 9: Normal Distribution is Symmetric with 68% within 1σ and 95% within 2σ

An alternative approach to using the familiar normal distribution area breakdown in the graph above is to use the standard normal distribution and the `normalCdf()` function to find the area between -1 and 2. Remember that in the standard normal distribution, the mean is zero and the standard deviation is one. So, the probability area between one standard deviation below the mean and two standard deviations above the mean is `normalCdf(-1,2,0,1)`$= 0.818595 \approx 0.819$ or as a percentage 81.9% to three significant figures.

Solution 9(b)

We're told that $P(L < 53.92) = 0.975$. Since we know an area amount, we can use the `invNorm()` inverse normal function to determine the corresponding z-value using the Standard Normal Distribution whose mean we always know is zero (0) and whose standard deviation we always know is 1.

Using our calculator, we compute `invNorm(0.975,0,1)` $= 1.95996$. This value is a z-value based on the standard normal distribution, so to convert it back to a data value in terms of the data set of nail lengths we are given, we need to use the z-score formula and then solve for the only unknown variable, which is the σ that we are looking for:

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ 1.95996 &= \frac{53.92-50}{\sigma} \\ \sigma &= \frac{3.92}{1.95996} \\ \sigma &= 2.00004 \\ \sigma &\approx 2.00 \quad \text{(3sf)} \end{aligned}$$

Solution 9(c)

All nails longer than t mm are considered "large" nails. We're told that the probability that a randomly selected nail is a large one is 0.75. The inverse normal function takes and area/probability and tells us the data value such that the probability of chosing an item less than or equal to that value gives the original probability provided. Here we're told that the probability 0.75 represents the chance of selecting a nail that is *longer than* t mm so we need to rephrase the problem in terms of a probability that is shorter than t in order to use the inverse normal function. If $P(L > t) = 0.75$ then we know that $P(L < t) = 1 - 0.75 = 0.25$. Using the inverse normal function with the mean and standard deviation that we now know, we can compute the answer: `invNorm(0.25,50,2)` $= 48.651$ So to three significant figures, the answer is $t \approx 48.7\,\text{mm}$.

Solution 9(d)(i)

We're told that a nail is selected at random from the large nails, and we're asked to find the probability that its length is less than 50.1 mm. So, this means we need to find:

$$P(L < 50.1 \,|\, L \,\text{is large}) = \frac{P(L < 50.1 \,\text{and}\, L \,\text{is large})\cdot}{P(L \,\text{is large})}$$

The probability in the numerator that $L < 50.1$ and L is large is the probability that the length L falls between 48.651 and 50.1 millimetres. We can use the `normalCdf()` function with these lower and upper bounds to compute the value for the numerator: `normalCdf(48.651,50.1,50,2)=0.269942`. Putting this value in the numerator and using the probability of 0.75 for a nail's being large in the denominator, we get the final answer:

$$P(L < 50.1 \,|\, L \,\text{is large}) = \frac{P(L < 50.1 \,\text{and}\, L \,\text{is large})\cdot}{P(L \,\text{is large})} = \frac{0.269942}{0.75} = 0.359923$$

Therefore, to three significant figures the answer is 0.360

Solution 9(d)(ii)

In the context of this problem, where we are choosing ten random nails that we already know are large, we're looking at a binomial distribution where the "success" case is picking a nail less than 50.1 mm whose probability we determined in the previous step as 0.36. So, we are asked to compute the probability that at least two nails (out of the ten) have a length that is less than 50.1. That is, we need to compute the probability of having at least two successes out of ten trials, given that the probability of success is 0.360

Using the `binomCdf()` with a low value of 2 and an upper value of 10, we can compute the desired answer: `binomCdf(10,0.36,2,10)=0.923619`

If your using a calculator whose `binomCdf()` does not support using a lower bound, then you can compute:

$$\texttt{binomCdf(10,0.36,10)} - \texttt{binomCdf(10,0.36,1)} = 0.923619$$

Therefore, to three significant figures, the answer is 0.924.

Problem 10

Solution 10(a)

Side $OA = r$ and side $OB = r$ as well since they are both radii of the same circle with center at O. Using the Law of Cosines, we can express the length of the side opposite angle θ using:

$$(AB)^2 = r^2 + r^2 - 2r \cdot r \cos\theta$$
$$(AB)^2 = 2r^2 - 2r^2 \cos\theta$$
$$(AB)^2 = 2r^2 (1 - \cos\theta)$$

Since we're told $ABCD$ is a square, its area is $(AB)^2$ which we showed above is $2r^2 (1 - \cos\theta)$.

Solution 10(b)(i)

When $\theta = \alpha$ the area of the square $ABCD$ equals the area of the sector OAB. The area of the sector is $\dfrac{\alpha}{2}r^2$

Solution 10(b)(ii)

If the area of the square equals the area of the sector, then we have:

$$\begin{aligned} 2r^2(1-\cos\alpha) &= \frac{\alpha}{2}r^2 \\ 2(1-\cos\alpha) &= \frac{\alpha}{2} \\ 4(1-\cos\alpha) &= \alpha \\ 4-4\cos\alpha &= \alpha \\ 4-4\cos\alpha-\alpha &= 0 \end{aligned}$$

With no obvious technique to solve the equation for α, we resort to graphing the equation and finding the zeros using the "Analyze Graph" feature as shown in Figure 10.

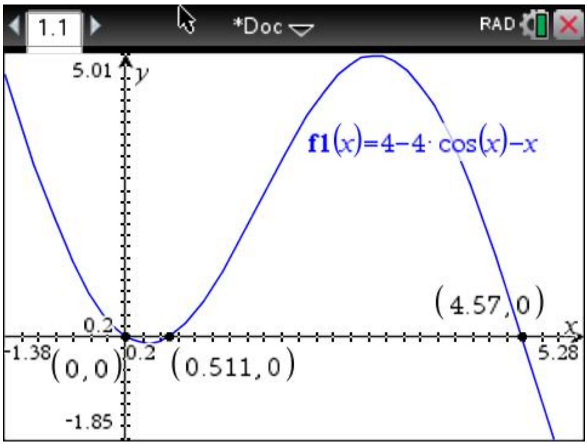


Figure 10: Graphing to Find Zeros to Solve Complex Equation

On the graph, we have identified three zeros at $x = 0$, $x = 0.511$, and $x = 5.28$, however we can rule out the first and the last ones because we are told in the problem that the original angle θ was strictly between 0.5 and π. Therefore, the only value of α that satisfies the constraints of the problem is $\alpha = 0.511$ radians.

Solution 10(c)

When $\theta = \beta$, the area of R is more than twice the area of the sector. We are asked to find all possible values of β that make the following inequality true:

$$\begin{aligned}
\text{Area of } R &> 2\,(\text{Area of Sector } OAB) \\
(\text{Area of Square}) - (\text{Area of Segment}) &> 2\,(\text{Area of Sector } OAB) \\
(\text{Area of Square}) - ((\text{Area of Sector } OAB) - (\text{Area of Triangle})) &> 2\,(\text{Area of Sector } OAB) \\
(\text{Area of Square}) - 3\,(\text{Area of Sector } OAB) + (\text{Area of Triangle}) &> 0 \\
2r^2\,(1-\cos\beta) - 3\left(\frac{\beta}{2}r^2\right) + \frac{1}{2}r^2\sin\beta &> 0 \\
\frac{r^2}{2}\,(4 - 4\cos\beta - 3\beta + \sin\beta) &> 0 \\
4 - 4\cos\beta - 3\beta + \sin\beta &> 0
\end{aligned}$$

Again, we can use the graphing calculator to find all the values of β between 0.5 and π that make this expression greater than zero. It will be all the places on this interval where the graph is above the x-axis that fall within the possible range $0.5 < \beta < \pi$ that the angle can have. As shown in Figure 11, we've found a zero using "Analyze Graph" at $x = 1.31$ and another zero at 2.67. We ignore the third zero at the origin because it lies outside of the range of 0.5 to π. Therefore, over the range of 0.5 to π the graph is above the x-axis for values of $1.31 < \beta < 2.67$, so these are the values of β for which the area of R is more than twice the area of the sector.

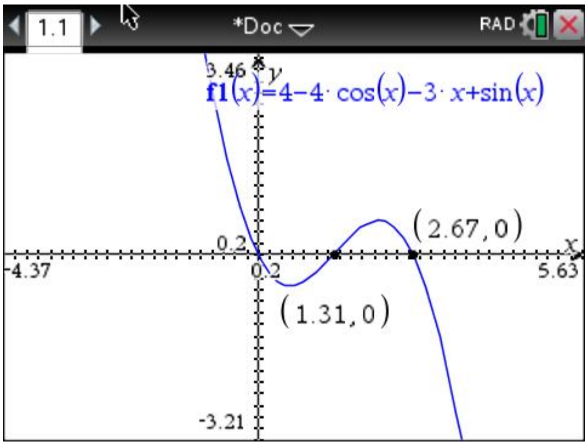


Figure 11: Graphing to Find Solution to Complex Inequality