IB Mathematics SL

Study and Revision Guide
(with questions from the May 2014 examination)

Paper 1:	Calculator Inactive	(90 marks/90 min)
Paper 2:	Calculator Active	(90 marks/90 min)

Topic 1: Algebra

- 1.1 Sequences and series
- 1.2 Exponents and logarithms
- 1.3 Binomial theorem

Arithmetic Sequences and Series

Formulas

- $u_n = u_1 + (n-1)d$
- $S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$

A sequence is **arithmetic** if there is a **common difference** d such that $u_{n+1} = u_n + d$ for all $n \in \mathbb{Z}^+$.

To show that a sequence is arithmetic, you must show that $u_{n+1} - u_n = d$.

Geometric Sequences and Series

Formulas

- $u_n = u_1 r^{n-1}$
- $S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r}, r \neq 1$
- $S_\infty = \frac{u_1}{1-r}, |r| < 1$

A sequence is **geometric** if there is a **common ratio** r such that $u_{n+1} = ru_n$ for all $n \in \mathbb{Z}^+$.

To show that a sequence is geometric, you must show that $\dfrac{u_{n+1}}{u_n} = r$.

Compound Interest Problems

Memorize This!

- $A = P(1 + \frac{r}{n})^{nt}$
- A: future amount of the investment
- P: initial amount of the investment
- n: number of compounding periods per year
- r: interest rate (in decimal form)
- t: number of years since the initial investment was made

Exam Practice (14M.1.SL.TZ1.2)

In an arithmetic sequence, the third term is 10 and the fifth term is 16.

1. Find the common difference.
2. Find the first term.
3. Find the sum of the first 20 terms of the sequence.

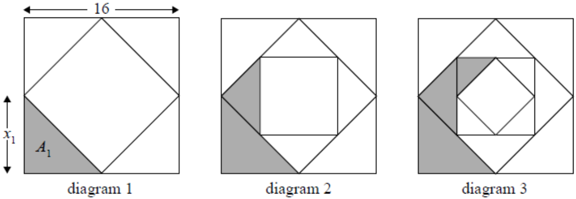
Exam Practice (14M.1.SL.TZ1.2)

In an arithmetic sequence, the third term is 10 and the fifth term is 16.

1. Find the common difference. $d = 3$
2. Find the first term. $u_1 = 4$
3. Find the sum of the first 20 terms of the sequence. $S_{20} = 650$

Exam Practice (14M.1.SL.TZ1.10)

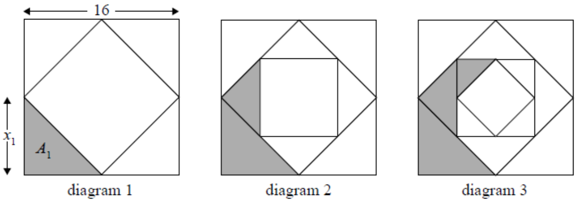
The sides of a square are 16 cm in length. The midpoints of the sides of this square are joined to form a new square and four triangles (diagram 1). The process is repeated twice, as shown in diagrams 2 and 3.



Let x_n denote the length of one of the equal sides of each new triangle.

Let A_n denote the area of each new triangle.

Exam Practice (14M.1.SL.TZ1.10)



1. Complete the table.

n	1	2	3
x_n	8	$\sqrt{32}$	4
A_n	32	16	8

2. The process is repeated. Find A_6. $A_6 = 1$

3. Consider an initial square of side length k cm. The process is repeated indefinitely. The total area of the shaded region is k cm^2. Find k. $k = 4$

Sequences and Series

Additional Practice with Sequences and Series

Haese Mathematics SL (3rd Edition)

Chapter 25A #1, 2, 28, 36, 42, 62

Chapter 25B #1, 4, 5, 22, 69

Laws of Exponents

Memorize This!

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $a^{xy} = (a^x)^y = (a^y)^x$
- $(ab)^x = a^x b^x$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

- $e^{x+y} =$
- $e^{x-y} =$
- $e^{xy} =$
- $(2e)^x =$
- $\left(\frac{2}{e}\right)^x =$

Laws of Exponents

Memorize This!

- $a^{x+y} = a^x a^y$
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- $e^{x+y} = e^x e^y$
- $e^{x-y} = \frac{e^x}{e^y}$
- $e^{xy} = (e^x)^y = (e^y)^x$
- $(2e)^x = 2^x e^x$
- $\left(\frac{2}{e}\right)^x = \frac{2^x}{e^x}$

Laws of Logarithms

Formulas

- $a^x = b \Leftrightarrow x = \log_a b$
- $a^x = e^{x \ln a}$
- $\log_a a^x = x = a^{\log_a x}$

- $2^x = 5 \Leftrightarrow$
- $2^x =$
- $\log_3 3^x =$

Laws of Logarithms

Formulas

$a^x = b \Leftrightarrow x = \log_a b$	$2^x = 5 \Leftrightarrow x = \log_2 5$
$a^x = e^{x \ln a}$	$2^x = e^{x \ln 2}$
$\log_a a^x = x = a^{\log_a x}$	$\log_3 3^x = x = 3^{\log_3 x}$

Laws of Logarithms

Formulas

- $\log_c a + \log_c b = \log_c ab$
- $\log_c a - \log_c b = \log_c \frac{a}{b}$
- $\log_c a^r = r \log_c a$
- $\log_b a = \frac{\log_c a}{\log_c b}$

- $\log_7 2 + \log_7 3 =$
- $\log 12 - \log 3 =$
- $\ln 2^5 =$
- $\log_3 4 =$

Laws of Logarithms

Formulas

- $\log_c a + \log_c b = \log_c ab$
- $\log_c a - \log_c b = \log_c \frac{a}{b}$
- $\log_c a^r = r \log_c a$
- $\log_b a = \frac{\log_c a}{\log_c b}$

- $\log_7 2 + \log_7 3 = \log_7 6$
- $\log 12 - \log 3 = \log 4$
- $\ln 2^5 = 5 \ln 2$
- $\log_3 4 = \frac{\log 4}{\log 3} = \frac{\ln 4}{\ln 3}$

Exam Practice (14M.1.SL.TZ1.4)

Write down the value of:

1. $\log_3 27$
2. $\log_8 \frac{1}{8}$
3. $\log_{16} 4$

Hence, solve $\log_3 27 + \log_8 \frac{1}{8} - \log_{16} 4 = \log_4 x$

Exam Practice (14M.1.SL.TZ1.4)

Write down the value of:

1. $\log_3 27$ 3
2. $\log_8 \frac{1}{8}$ -1
3. $\log_{16} 4$ $\frac{1}{2}$

Hence, solve $\log_3 27 + \log_8 \frac{1}{8} - \log_{16} 4 = \log_4 x$ $x = 8$

Exponents and Logarithms

Additional Practice with Exponents and Logarithms

Haese Mathematics SL (3rd Edition)

Chapter 25A #23, 52, 60, 76, 81, 85

Chapter 25B #31, 57, 59

Binomial Theorem

Formulas

- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + b^n$

$$(2x+3)^4 =$$

$$=$$

Binomial Theorem

Formulas

- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + b^n$

$$
\begin{aligned}
(2x+3)^4 &= 1(2x)^4 + 4(2x)^3(3) + 6(2x)^2(3)^2 + 4(2x)(3)^3 + 1(3)^4 \\
&= 16x^4 + 96x^3 + 216x^2 + 216x + 81
\end{aligned}
$$

Pascal's Triangle

The coefficients of the expansion of $(a+b)^n$ can also be found using Pascal's triangle.

$$
\begin{array}{lccccccccccccccl}
n=0 & & & & & & & 1 & & & & & & & \leftarrow \text{row } 0 \\
n=1 & & & & & & 1 & & 1 & & & & & & \leftarrow \text{row } 1 \\
n=2 & & & & & 1 & & 2 & & 1 & & & & & \leftarrow \text{row } 2 \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & & \leftarrow \text{row } 3 \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \leftarrow \text{row } 4 \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \leftarrow \text{row } 5 \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \leftarrow \text{row } 6
\end{array}
$$

Example:

$\binom{3}{1} =$ $\qquad$ $\binom{4}{2} =$ $\qquad$ $\binom{6}{3} =$

Pascal's Triangle

The coefficients of the expansion of $(a+b)^n$ can also be found using Pascal's triangle.

$$\begin{array}{lccccccccccccccl}
n=0 & & & & & & & 1 & & & & & & & \leftarrow \text{row } 0 \\
n=1 & & & & & & 1 & & 1 & & & & & & \leftarrow \text{row } 1 \\
n=2 & & & & & 1 & & 2 & & 1 & & & & & \leftarrow \text{row } 2 \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & & \leftarrow \text{row } 3 \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \leftarrow \text{row } 4 \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \leftarrow \text{row } 5 \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \leftarrow \text{row } 6
\end{array}$$

Example:

$\binom{3}{1} = 3$ $\qquad$ $\binom{4}{2} = 6$ $\qquad$ $\binom{6}{3} = 20$

Exam Practice (14M.2.SL.TZ1.2)

Consider the expansion of $(x+3)^{10}$.

1. Write down the number of terms in this expansion.
2. Find the term containing x^3.

Exam Practice (14M.2.SL.TZ1.2)

Consider the expansion of $(x+3)^{10}$.

1. Write down the number of terms in this expansion. 11 terms
2. Find the term containing x^3. $262440x^3$

Binomial Theorem

Additional Practice with the Binomial Theorem

Haese Mathematics SL (3rd Edition)

Chapter 25A #5, 29, 86, 87

Chapter 25B #3, 16

Topic 2: Functions and Equations

- 2.1 Functions in general
- 2.2 Transformations of graphs
- 2.3 Basic functions
- 2.4 Equation solving

Functions in General

- A relation is called a **function** if each $x \in D$ corresponds to a unique $y \in R$
- In order to be a function, the graph of $y = f(x)$ must pass the **vertical line test**

Function Notation

$f(x) = \sqrt{x+1}$	$f : x \mapsto \sqrt{x+1}$

One-to-One Functions

- A function is **one-to-one** if each $x \in D$ corresponds to a unique $y \in R$ and each $y \in R$ corresponds to a unique $x \in D$
  - A one-to-one function has an **inverse function**
  - A one-to-one function must pass BOTH the **vertical line test** and the **horizontal line test**
- If f is not one-to-one, it is called **many-to-one**
  - A many-to-one function will not have an inverse function unless we restrict the domain (e.g., $f(x) = \cos x, x \in [0, \pi]$)

Inverse Functions

- If f^{-1} is the **inverse function** of f, and $f(a) = b$, then $f^{-1}(b) = a$.
- The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.
- Both $f \circ f^{-1}$ and $f^{-1} \circ f$ are **identity functions** since
  - $(f \circ f^{-1})(x) = x$
  - $(f^{-1} \circ f)(x) = x$
- If $f = f^{-1}$, then f is called **self-inverse**.

Even and Odd Functions

Even and Odd Functions

- A function f is even iff $f(-x) = f(x)$ for all x in the domain.
- Even functions are symmetric about the y-axis.

- A function f is odd iff $f(-x) = -f(x)$ for all x in the domain.
- Odd functions are symmetric about the origin.

Graph Sketching

Key features to include when drawing/sketching a graph:

- Domain
- x- and y-intercepts
- Endpoint values (if applicable)
- Vertical and horizontal asymptotes
- Local maxima and minima
- Points of inflection

Exam Practice (14M.2.SL.TZ1.10)

Let $f(x) = \frac{3x}{x-q}$, where $x \neq q$.

1. Write down the equations of the vertical and horizontal asymptotes of the graph of f.
2. The vertical and horizontal asymptotes to the graph of f intersect at the point $Q(1, 3)$. Find the value of q.
3. The point $P(x, y)$ lies on the graph of f. Show that $PQ = \sqrt{(x-1)^2 + \left(\frac{3}{x-1}\right)^2}$.
4. Find the coordinates of the points on the graph of f that are closest to $(1, 3)$.

Exam Practice (14M.2.SL.TZ1.10)

Let $f(x) = \frac{3x}{x-q}$, where $x \neq q$.

1. Write down the equations of the vertical and horizontal asymptotes of the graph of f. $x = q, y = 3$
2. The vertical and horizontal asymptotes to the graph of f intersect at the point $Q(1, 3)$. Find the value of q. $q = 1$
3. The point $P(x, y)$ lies on the graph of f. Show that $PQ = \sqrt{(x-1)^2 + \left(\frac{3}{x-1}\right)^2}$.
4. Find the coordinates of the points on the graph of f that are closest to $(1, 3)$. $(-0.732, 1.27), (2.73, 4.72)$

Binomial Theorem

Additional Practice with Functions in General

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Chapter 25A #3, 6, 11, 34, 32, 34, 39, 46a-c, 64, 83

Chapter 25B #48

Translations of Graphs

Translations

$y = f(x) + d$	vertical translation up d units
$y = f(x) - d$	vertical translation down d units
$y = f(x - c)$	horizontal translation right c units
$y = f(x + c)$	horizontal translation left c units

Translations are often described using vector notation. For example, translation by the vector $\binom{c}{d}$ causes a horizontal translation right c and a vertical translation up d.

Dilations of Graphs

Dilations

$y = af(x)$	vertical stretch by a scale factor of a
$y = f(bx)$	horizontal stretch by a scale factor of $\frac{1}{b}$

Describe the transformations from the parent function $f(x) = \sqrt{x}$ to the new function $g(x) = 2\sqrt{3x}$.

Dilations of Graphs

Dilations

$y = af(x)$	vertical stretch by a scale factor of a
$y = f(bx)$	horizontal stretch by a scale factor of $\frac{1}{b}$

Describe the transformations from the parent function $f(x) = \sqrt{x}$ to the new function $g(x) = 2\sqrt{3x}$.

Vertical stretch by a factor of 2; horizontal stretch by a factor of $\frac{1}{3}$

Reflections of Graphs

Reflections

$y = -f(x)$	vertical reflection about the x-axis
$y = f(-x)$	horizontal reflection about the y-axis

Horizontal Transformations of Graphs

Horizontal Transformations

$$y = f(-b(x-c))$$

Changes to the *input* of a function will produce **horizontal** transformations.

Order of Horizontal Transformations:

1. Horizontal Translation
2. Horizontal Dilation
3. Horizontal Reflection

Vertical Transformations of Graphs

Vertical Transformations

$y = -af(x) + d$	Changes to the *output* of a function will produce **vertical** transformations.

Order of Vertical Transformations:

1. Vertical Dilation
2. Vertical Reflection
3. Vertical Translation

Transformations of Graphs

Composite Transformations

$$y = -af(-b(x-c)) + d$$

Order of Vertical:

1. Vertical Dilation
2. Vertical Reflection
3. Vertical Translation

Order of Horizontal:

1. Horizontal Translation
2. Horizontal Dilation
3. Horizontal Reflection

TIP: The vertical and horizontal transformations can be applied independently of one another. However, it's easiest to apply all the verticals first, then apply all the horizontals.

Binomial Theorem

Additional Practice with Transformations of Graphs

Haese Mathematics SL (3rd Edition)

Chapter 25A #4, 51, 78

Linear Functions

Memorize This!

$y = ax + b$	slope-intercept form
$Ax + By = C$	standard form
$y - y_1 = m(x - x_1)$	point-slope form

The slope of a function is called the **gradient**.

If two lines are parallel, they have equal gradients.

If two lines are perpendicular, the product of their gradients is -1.

Quadratic Functions

Formulas

- $f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry $x = -\frac{b}{2a}$
- $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$

If you need to rewrite a quadratic equation from standard form to vertex form, you must either:

1. complete the square
2. use the axis of symmetry to find the x-coordinate of the vertex, then use substitution to find the y-coordinate of the vertex

Rational Functions

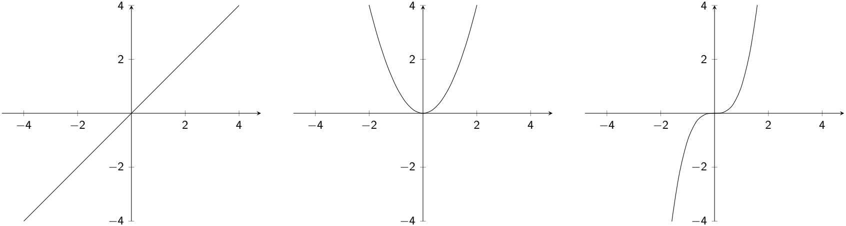
Let $f(x) = \frac{ax+b}{cx+d}, c \neq 0$

- f has a vertical asymptote at $x = -\frac{d}{c}$
- f has a horizontal asymptote at $y = \frac{a}{c}$
- f has a x-intercept at $\left(-\frac{b}{a}, 0\right)$ if $a \neq 0$
- f has a y-intercept at $\left(0, \frac{b}{d}\right)$ if $d \neq 0$

Any value of x that makes the numerator and denominator simultaneously equal to zero produces a hole (removable discontinuity) in the graph.

Graphs of Basic Functions

Write down the equation for each of the following functions.



Graphs of Basic Functions

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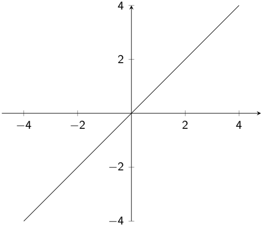
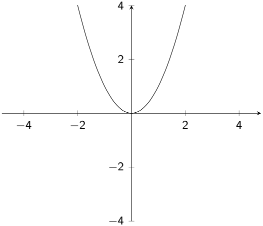
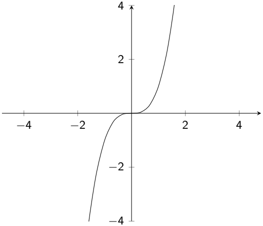


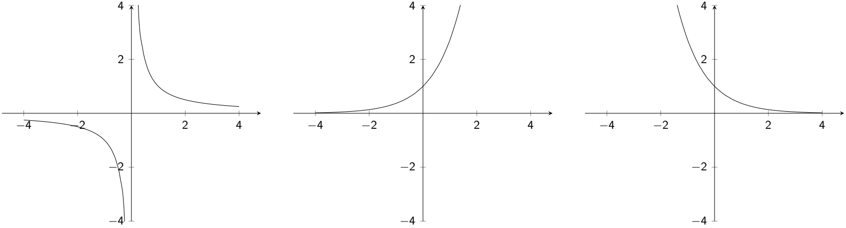




$y = x$ $\qquad$ $y = x^2$ $\qquad$ $y = x^3$

Graphs of Basic Functions

Write down the equation for each of the following functions.



Graphs of Basic Functions

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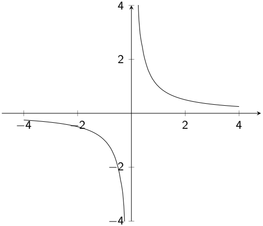
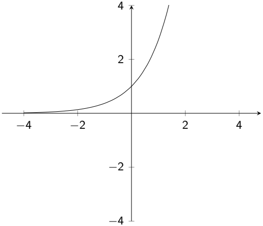
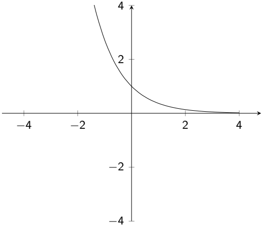


$y = \frac{1}{x}$



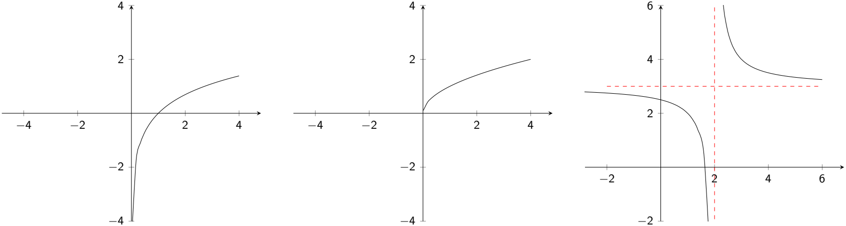
$y = e^x$



$y = e^{-x}$

Graphs of Basic Functions

Write down the equation for each of the following functions.



Graphs of Basic Functions

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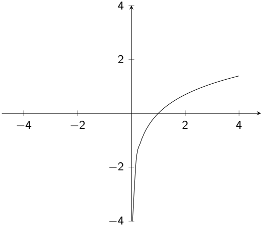
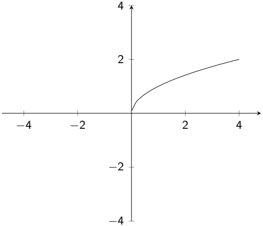
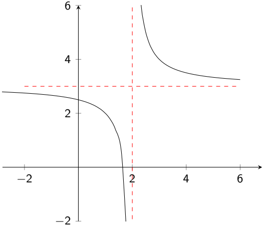


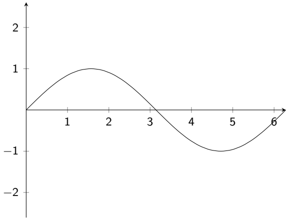


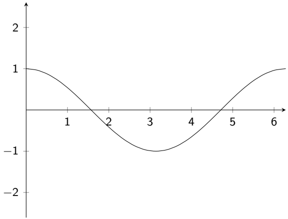


$y = \ln x$ $\qquad$ $y = \sqrt{x}$ $\qquad$ $y = \frac{1}{x-2} + 3$

Graphs of Basic Functions

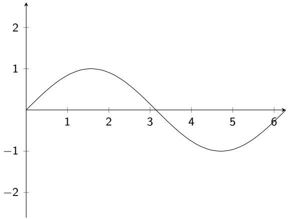
Write down the equation for each of the following functions.

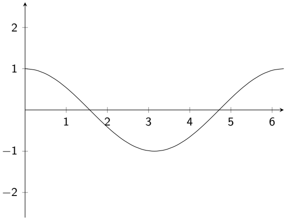




Graphs of Basic Functions

Write down the equation for each of the following functions.





$y = \sin x$

$y = \cos x$

Exam Practice (14M.1.SL.TZ1.1)

Let $f(x) = a(x-h)^2 + k$. The vertex of the graph of f is at $(2,3)$ and the graph passes through $(1,7)$.

1. Write down the value of h and of k.
2. Find the value of a.

Exam Practice (14M.1.SL.TZ1.1)

Let $f(x) = a(x-h)^2 + k$. The vertex of the graph of f is at $(2,3)$ and the graph passes through $(1,7)$.

1. Write down the value of h and of k. $h = 2, k = 3$
2. Find the value of a. $a = 4$

The Discriminant

Formulas

$$\Delta = b^2 - 4ac$$

Let $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$

- If $\Delta = 0$, then $f(x) = 0$ has one solution (two equal roots)
- If $\Delta > 0$, then $f(x) = 0$ has two solutions (two distinct roots)
- If $\Delta < 0$, then $f(x) = 0$ has no solution (no real roots)

Exam Practice (14M.1.SL.TZ1.7)

Let $f(x) = px^3 + px^2 + qx$.

1. Find $f'(x)$.
2. Given that $f'(x) \geq 0$, show that $p^2 \leq 3pq$.

Exam Practice (14M.1.SL.TZ1.7)

Let $f(x) = px^3 + px^2 + qx$.

1. Find $f'(x)$. $f'(x) = 3px^2 + 2px + q$
2. Given that $f'(x) \geq 0$, show that $p^2 \leq 3pq$.

   Since $f'(x) \geq 0$, $f'(x) = 0$ has either one real root (two equal roots) or no real roots.

   Hence, $\Delta = 4(p^2 - 3pq) \leq 0 \Rightarrow p^2 \leq 3pq$

Quadratic Form

Remember!

Some equations that are not quadratic can be rewritten in quadratic form.

Rewrite the following in quadratic form, then solve.

1. $4^x + 2^x = 6$
2. $x^4 - 2x^2 + 1 = 0$

Quadratic Form

Remember!

Some equations that are not quadratic can be rewritten in quadratic form.

Rewrite the following in quadratic form, then solve.

1. $4^x + 2^x = 6$ $(2^x)^2 + (2^x) - 6 = 0; x = 1$
2. $x^4 - 2x^2 + 1 = 0$ $(x^2)^2 + 2(x^2) + 1 = 0; x = \pm 1$

Equation Solving

Additional Practice with Equation Solving

Haese Mathematics SL (3rd Edition)

Chapter 25A #9, 59a

Chapter 25B #2a-c, 12, 33, 39, 46a-c

Topic 3: Circular Functions and Trigonometry

Topic 3: Circular Functions and Trigonometry

- 3.1 Radian measure
- 3.2 Circular functions
- 3.3 Trigonometric equations
- 3.4 Triangle trigonometry

Circles

Formulas

- **Length of an arc:** $l = \theta r$
- **Area of a sector:** $A = \frac{1}{2}\theta r^2$

Remember: The above formulas use **radian** measure.

1. To convert an angle from degree measure to radian measure, multiply by $\frac{\pi}{180^\circ}$.
2. To convert an angle from radian measure to degree measure, multiply by $\frac{180^\circ}{\pi}$.

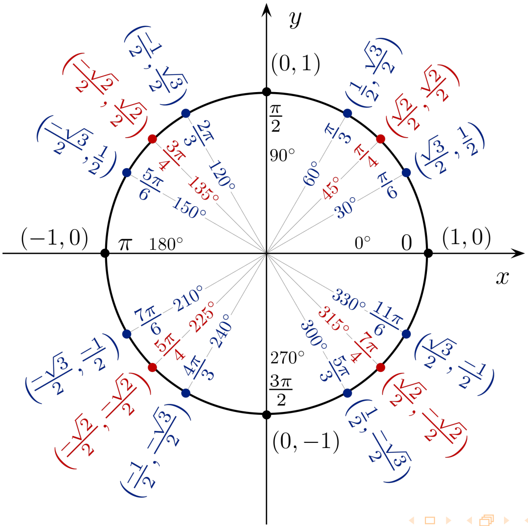
Trigonometric Identities

Formulas

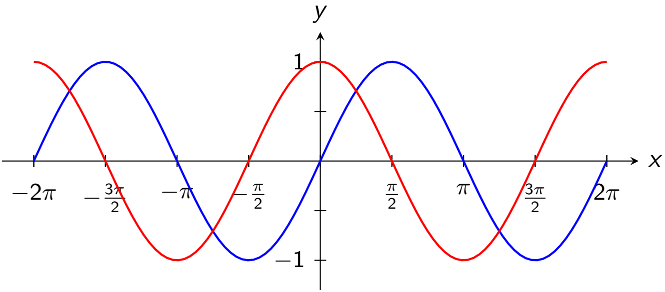
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cos^2\theta + \sin^2\theta = 1$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

Remember: All points on the unit circle have coordinates of the form $(\cos\theta, \sin\theta)$, where θ is the angle measured from the positive x-axis.

The Unit Circle



The Sine and Cosine Functions

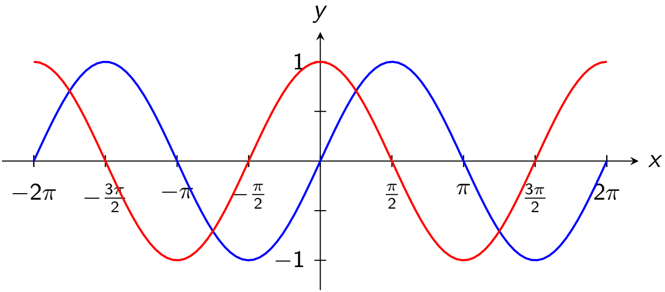


Domain:

Range:

Period:

The Sine and Cosine Functions

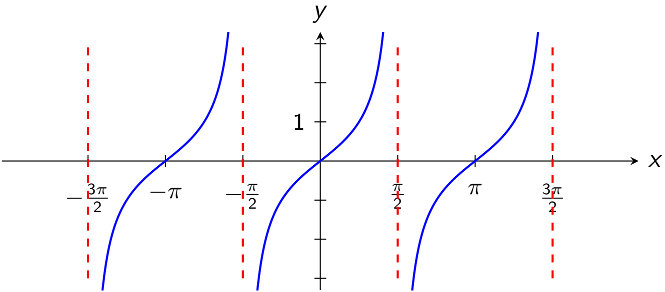


Domain: $x \in \mathbb{R}$

Range: $-1 \leq y \leq 1$

Period: 2π

The Tangent Function

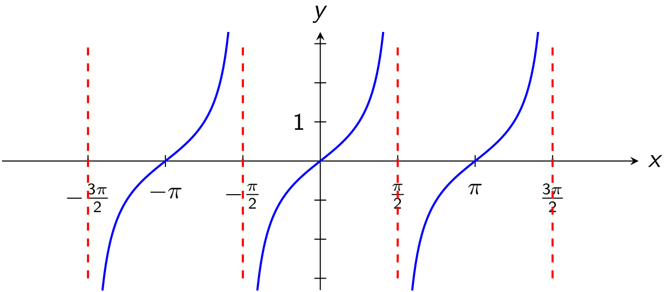


Domain:

Range:

Period:

The Tangent Function



Domain: $x \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

Range: $y \in \mathbb{R}$

Period: π

Trigonometric Identities

Memorize This!

- $\sin(x + 2\pi) =$
- $\cos(x + 2\pi) =$
- $\tan(x + \pi) =$

Remember, for a **periodic function**, adding the period to x simply takes you to the corresponding point in the next cycle, so it has the same y-value.

Trigonometric Identities

Memorize This!

- $\sin(x + 2\pi) = \sin x$
- $\cos(x + 2\pi) = \cos x$
- $\tan(x + \pi) = \tan x$

Remember, for a **periodic function**, adding the period to x simply takes you to the corresponding point in the next cycle, so it has the same y-value.

Trigonometric Identities

Memorize This!

- $\sin(-x) =$
- $\cos(-x) =$
- $\tan(-x) =$

- Hence, sine is an **odd** function.
- Hence, cosine is an **even** function.
- Hence, tangent is an **odd** function.

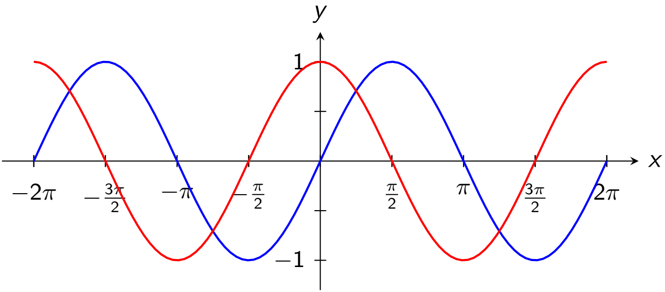
Trigonometric Identities

Memorize This!

- $\sin(-x) = -\sin(x)$
- $\cos(-x) = \cos(x)$
- $\tan(-x) = \tan(x)$

- Hence, sine is an **odd** function.
- Hence, cosine is an **even** function.
- Hence, tangent is an **odd** function.

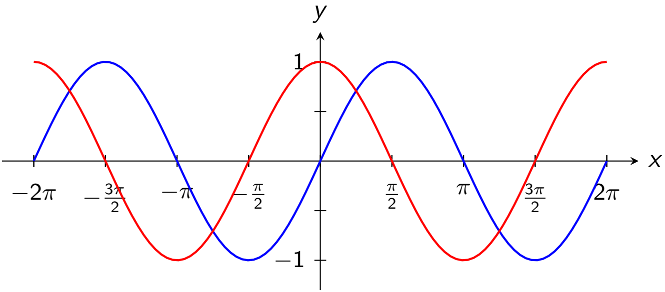
The Sine and Cosine Functions



$\sin(x + \frac{\pi}{2}) =$ $\quad$ $\sin(x - \frac{\pi}{2}) =$ $\quad$ $\sin(\frac{\pi}{2} - x) =$

$\cos(x + \frac{\pi}{2}) =$ $\quad$ $\cos(x - \frac{\pi}{2}) =$ $\quad$ $\cos(\frac{\pi}{2} - x) =$

The Sine and Cosine Functions

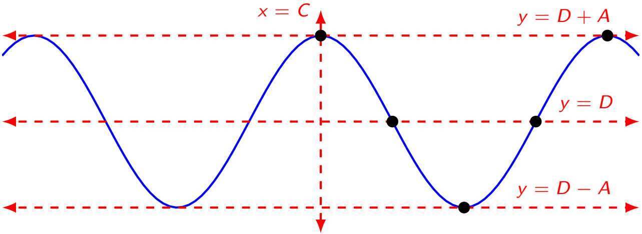


$\sin(x+\frac{\pi}{2}) = \cos x$ $\quad$ $\sin(x-\frac{\pi}{2}) = -\cos x$ $\quad$ $\sin(\frac{\pi}{2}-x) = \cos x$

$\cos(x+\frac{\pi}{2}) = -\sin x$ $\quad$ $\cos(x-\frac{\pi}{2}) = \sin x$ $\quad$ $\cos(\frac{\pi}{2}-x) = \sin x$

Transformations of the Cosine Function

$$y = A\cos(B(x - C)) + D$$

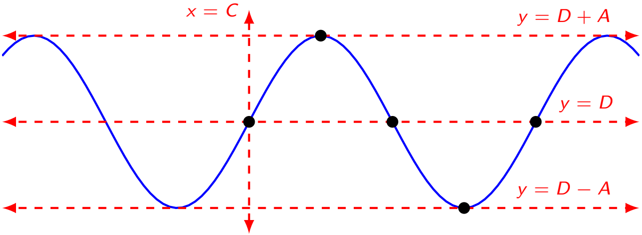


$$\text{period} = \frac{2\pi}{|B|}$$

To move from one key point to the next, add one-fourth the period to the x-coordinate.

Transformations of the Sine Function

$$y = A\sin(B(x - C)) + D$$



$$\text{period} = \frac{2\pi}{|B|}$$

To move from one key point to the next, add one-fourth the period to the x-coordinate.

Transformations of Circular Functions

A sine function has a minimum point at $(6, 0.3)$ and the nearest maximum point is at $(10, 3.3)$.

1. Write a sine equation to model the function.
2. Write a cosine equation to model the function.

Transformations of Circular Functions

A sine function has a minimum point at $(6, 0.3)$ and the nearest maximum point is at $(10, 3.3)$.

1. Write a sine equation to model the function.

   $A = \frac{3.3-0.3}{2}$; $B = \frac{2\pi}{2(10-6)}$; $C = \frac{6+10}{2}$; $D = \frac{3.3+0.3}{2}$

   $y = \frac{3}{2}\sin(\frac{\pi}{4}(x-8)) + \frac{4}{5}$

2. Write a cosine equation to model the function.

   $y = \frac{3}{2}\cos(\frac{\pi}{4}(x-10)) + \frac{4}{5}$

Circular Functions

Additional Practice with Circular Functions

Haese Mathematics SL (3rd Edition)

Chapter 25A #7, 10, 35, 79

Chapter 25B #10, 66, 67

Solving Trigonometric Equations

Reminders!

- Trig equations often have infinitely many solutions. Adding the period to one solution will give you another solution (in the next cycle). Take care to find **ONLY** and **ALL** solutions in the stated interval.
- **Never divide by zero!** When solving a trig equation, never divide both sides by $\sin x$, $\cos x$, or $\tan x$ if they could possibly equal zero. Instead, rearrange terms so the right-hand side equals zero, then factor.

Solving Trigonometric Equations

Solve $\sqrt{3}\cos x = \sin 2x$ on $[-\pi, \pi]$.

Solving Trigonometric Equations

Solve $\sqrt{3}\cos x = \sin 2x$ on $[-\pi, \pi]$.

$$\begin{aligned}\sqrt{3}\cos x &= 2\sin x\cos x\\ \sqrt{3}\cos x - 2\sin x\cos x &= 0\\ \cos x(\sqrt{3} - 2\sin x) &= 0\end{aligned}$$

$\cos x = 0$ or $\sqrt{3} - 2\sin x = 0$

$x = \pm\frac{\pi}{2}$ or $x = \frac{\pi}{3}, \frac{2\pi}{3}$

Solving Trigonometric Equations

Solve $1 + \cos x = 2\sin^2 x$ on $[0, 2\pi)$.

Solving Trigonometric Equations

Solve $1 + \cos x = 2\sin^2 x$ on $[0, 2\pi)$.

$$\begin{aligned} 1 + \cos x - 2(1 - \cos^2 x) &= 0 \\ 2\cos^2 x + \cos x - 1 &= 0 \\ 2w^2 + w - 1 &= 0 \\ (2w - 1)(w + 1) &= 0 \end{aligned}$$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$

Exam Practice (14M.2.SL.TZ1.9)

Let $f(x) = \cos(\frac{\pi}{4}x) + \sin(\frac{\pi}{4}x)$, for $-4 \leq x \leq 4$.

- Sketch the graph of f.
- Find the values of x where the function is decreasing.
- The function f can also be written in the form $f(x) = a\sin(\frac{\pi}{4}(x + c))$, where $a \in \mathbb{R}$, and $0 \leq c \leq 2$. Find the value of a.
- The function f can also be written in the form $f(x) = a\sin(\frac{\pi}{4}(x + c))$, where $a \in \mathbb{R}$, and $0 \leq c \leq 2$. Find the value of c.

Exam Practice (14M.2.SL.TZ1.9)

Let $f(x) = \cos(\frac{\pi}{4}x) + \sin(\frac{\pi}{4}x)$, for $-4 \leq x \leq 4$.

- Sketch the graph of f.
- Find the values of x where the function is decreasing.
  $-4 \leq x < -3; 1 < x \leq 4$
- The function f can also be written in the form $f(x) = a\sin(\frac{\pi}{4}(x + c))$, where $a \in \mathbb{R}$, and $0 \leq c \leq 2$. Find the value of a. $a = \sqrt{2}$
- The function f can also be written in the form $f(x) = a\sin(\frac{\pi}{4}(x + c))$, where $a \in \mathbb{R}$, and $0 \leq c \leq 2$. Find the value of c. $c = 1$

Trigonometric Equations

Additional Practice with Trigonometric Equations

Haese Mathematics SL (3rd Edition)

Chapter 25A #8, 44, 61, 63, 71

Chapter 25B #42, 73

Sine and Cosine Rules

Formulas

- **Cosine rule:** $c^2 = a^2 + b^2 - 2ab\cos C$; $\cos C = \frac{a^2+b^2-c^2}{2ab}$
- **Sine rule:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Area of a triangle:** $A = \frac{1}{2}ab\sin C$

Use the Law of Cosines when you have **SSS** or **SAS**.

Use the Law of Sines when you have **ASA** or **SSA**.

Remember: SSA may result in an ambiguous case.

Triangle Trigonometry

Let ABC be a triangle with $m\angle A = 30^\circ$ and $b = 4$. Find c and $m\angle C$ if:

1. $a = 1$
2. $a = 2$
3. $a = 3$
4. $a = 4$

Don't forget to switch your calculator to **degree mode** for this problem.

Then don't forget to switch it back to **radian mode** for the remainder of the exam!

Triangle Trigonometry

Let ABC be a triangle with $m\angle A = 30°$ and $b = 4$. Find c and $m\angle C$ if:

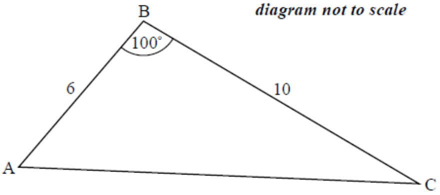
1. $a = 1$ no solution
2. $a = 2$ $c = 2; m\angle C = 60°$
3. $a = 3$ $c_1 = 2.67; m\angle C_1 = 41.8°; c_2 = 6.55; m\angle C_2 = 138.2°$
4. $a = 4$ $c = 6.93; m\angle C = 120°$

Don't forget to switch your calculator to **degree mode** for this problem.

Then don't forget to switch it back to **radian mode** for the remainder of the exam!

Exam Practice (14M.2.SL.TZ1.1)

The following diagram shows triangle ABC.

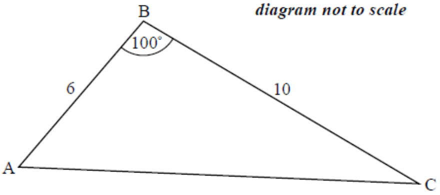


$\mathrm{AB} = 6\,\mathrm{cm}$, $\mathrm{BC} = 10\,\mathrm{cm}$, and $\mathrm{A\hat{B}C} = 100°$.

1. Find AC.
2. Find $B\hat{C}A$.

Exam Practice (14M.2.SL.TZ1.1)

The following diagram shows triangle ABC.



$AB = 6\text{ cm}$, $BC = 10\text{ cm}$, and $A\hat{B}C = 100^\circ$.

1. Find AC. $AC = 12.5$
2. Find $B\hat{C}A$. $B\hat{C}A = 28.2^\circ$

Triangle Trigonometry

Additional Practice with Triangle Trigonometry

Haese Mathematics SL (3rd Edition)

Chapter 25A #40, 45b, 58, 68

Chapter 25B #7, 54, 58, 62, 63

Topic 4: Vectors

- 4.1 Vector algebra
- 4.2 Equations of lines and planes

Vectors

- A **vector** is a quantity with both **magnitude** and **direction**. It is drawn as a directed line segment.
- The vector $\overrightarrow{AB}$ has **initial point** A and **terminal point** B.
- The vector $\overrightarrow{OA}$ has its initial point at the origin. Any such vector is called a **position vector**.
- $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is a position vector with **components** v_1, v_2, v_3.

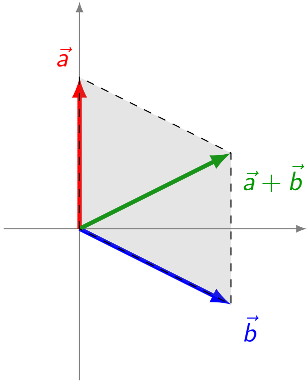
Vectors

Formulas

Magnitude of a vector: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

- Vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same magnitude but oppositie directions.
- Two vectors are equal if and only if they have the same **magnitude** and **direction**.
- A vector with magnitude zero is called the **zero vector** and is written $\vec{0}$.

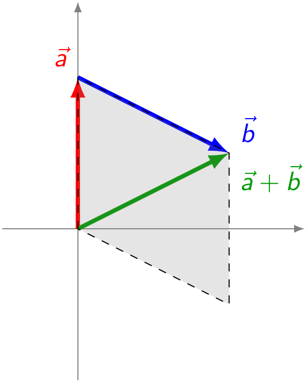
Vector Addition



To add two vectors graphically, either:

- place them tail-to-tail and draw the diagonal of the parallelogram formed by the two vectors, as shown
- place them head-to-tail and draw the vector from the initial point of the first to the terminal point of the second

Vector Addition



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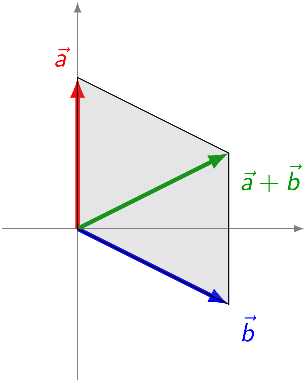
Vector Addition

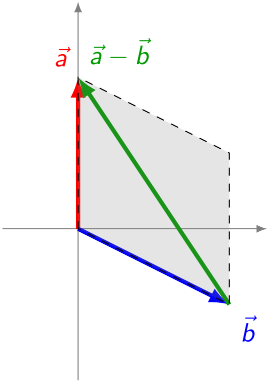


To add two or more vectors algebraically, simply add their corresponding components.

Let $\vec{a} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$a \vec{+} b = \begin{pmatrix} 0+2 \\ 2+(-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

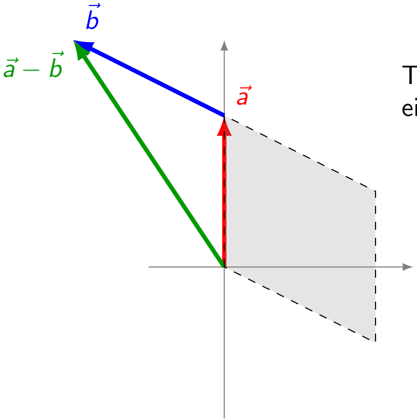
Vector Subtraction



To subtract two vectors graphically, either:

- place them tail-to-tail and draw the vector from the head of the second to the head of the first
- add the opposite of the second

Vector Subtraction



To subtract two vectors graphically, either:

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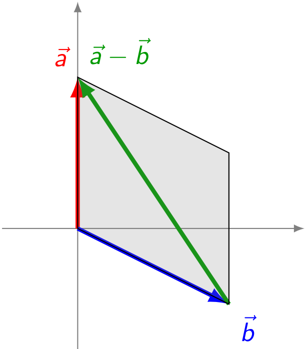
- add the opposite of the second

Vector Subtraction



To subtract two vectors algebraically, simply subtract their corresponding components.

Let $\vec{a} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$a \vec{+} b = \begin{pmatrix} 0 - 2 \\ 2 - (-1) \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Unit Vectors

A vector with magnitude 1 is called a **unit vector**.

Any non-zero vector can be rescaled so that it has a length of 1 (this is called normalizing a vector). To find a unit vector in the direction of $\vec{v}$, multiply the components of $\vec{v}$ by $\frac{1}{|\vec{v}|}$.

1. Find a unit vector in the direction of $\vec{v} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$
2. Find a unit vector in the direction of $\vec{w} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

Unit Vectors

A vector with magnitude 1 is called a **unit vector**.

Any non-zero vector can be rescaled so that it has a length of 1 (this is called normalizing a vector). To find a unit vector in the direction of $\vec{v}$, multiply the components of $\vec{v}$ by $\frac{1}{|\vec{v}|}$.

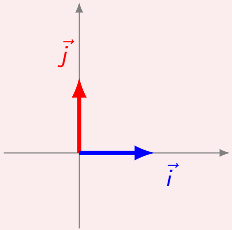
1. Find a unit vector in the direction of $\vec{v} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ $\vec{u} = \begin{pmatrix} \frac{3}{\sqrt{58}} \\ \frac{7}{\sqrt{58}} \end{pmatrix}$
2. Find a unit vector in the direction of $\vec{w} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ $\vec{u} = \begin{pmatrix} \frac{2}{\sqrt{21}} \\ \frac{-1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \end{pmatrix}$

Standard Unit Vectors

Memorize This!

The **standard unit vectors** in two dimensions are

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Any two-dimensional vector can be expressed in terms of $\vec{i}$ and $\vec{j}$.

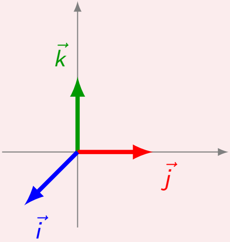
For example, $\begin{pmatrix} 3 \\ -2 \end{pmatrix} = 3\vec{i} - 2\vec{j}$.

Standard Unit Vectors

Memorize This!

The **standard unit vectors** in three dimensions are

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Any three-dimensional vector can be expressed in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

For example, $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3\vec{i} + 2\vec{j} + \vec{k}$.

Scalar (Dot) Product

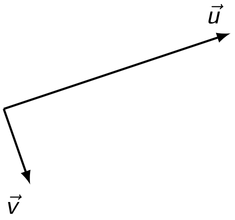
Formulas

- $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$
- $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
- $\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$

- The dot product can be used to find the angle between vectors.
- If two vectors $\vec{v}$ and $\vec{w}$ are **perpendicular**, then $\vec{v} \cdot \vec{w} = 0$.
- If two vectors $\vec{v}$ and $\vec{w}$ are **parallel**, then $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}|$ and $\vec{v} = k\vec{w}$, for some **scalar** k.

Exam Practice (14M.2.SL.TZ1.4)

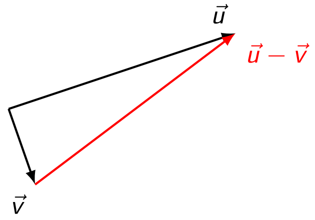
The following diagram shows two perpendicular vectors $\vec{u}$ and $\vec{v}$.



1. Let $\vec{w} = \vec{u} - \vec{v}$. Represent $\vec{w}$ on the diagram above.
2. Given that $\vec{u} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 5 \\ n \\ 3 \end{pmatrix}$, where $n \in \mathbb{Z}$, find n.

Exam Practice (14M.2.SL.TZ1.4)

The following diagram shows two perpendicular vectors $\vec{u}$ and $\vec{v}$.



1. Let $\vec{w} = \vec{u} - \vec{v}$. Represent $\vec{w}$ on the diagram above.
2. Given that $\vec{u} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 5 \\ n \\ 3 \end{pmatrix}$, where $n \in \mathbb{Z}$, find n. $n = -9$

Vector Algebra

Additional Practice with Vector Algebra

Haese Mathematics SL (3rd Edition)

Chapter 25A #12, 13

Chapter 25B #44, 72

Vector Equation of a Line

Formulas

$$\vec{r} = \vec{a} + t\vec{b}$$

Or, in component form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where $\vec{a}$ is the position vector of a point on the line, t is a scalar, and $\vec{b}$ is a direction vector.

Vector Equation of a Line

Write the vector equation of the line $y = 3x - 5$.

Vector Equation of a Line

Write the vector equation of the line $y = 3x - 5$.

Since the slope of the line is 3, we can use direction vector $\vec{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Since the point $(0, -5)$ lies on the line, we can use $\vec{a} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$.

Hence, the equation of the line is given by $\vec{r} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Lines in the Plane

Two lines in a plane are either:

- **parallel** (no points in common)
- **coincident** (all points in common)
- **intersecting** (one point in common)

Let $\vec{u}$ and $\vec{v}$ be the direction vectors for two lines in a plane.

- If $\vec{u} \parallel \vec{v}$ and the lines do not intersect, then they are **parallel**
- If $\vec{u} \parallel \vec{v}$ and the lines intersect, then they are **coincident**
- If $\vec{u} \nparallel \vec{v}$, then the lines are **intersecting**

Lines in 3D Space

Two lines in space are either:

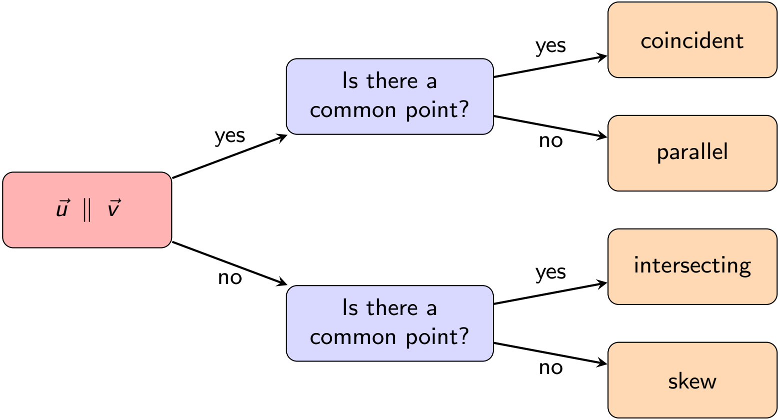
- **parallel** (coplanar but no points in common)
- **coincident** (coplanar and all points in common)
- **intersecting** (coplanar and one point in common)
- **skew** (not coplanar and no points in common)

Let $\vec{u}$ and $\vec{v}$ be the direction vectors for two lines in space.

- If $\vec{u} \parallel \vec{v}$ and the lines do not intersect, then they are **parallel**
- If $\vec{u} \parallel \vec{v}$ and the lines intersect, then they are **coincident**
- If $\vec{u} \nparallel \vec{v}$ and the lines intersect, then the lines are **intersecting**
- If $\vec{u} \nparallel \vec{v}$ and the lines do not intersect, then the lines are **skew**

Lines in 3D Space

Let $\vec{u}$ and $\vec{v}$ be the direction vectors for two lines in space.



Finding Points of Intersection

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + s \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

be the vector equations of two lines in 3D space.

To find points of intersection, we set the two equations equal to each other:

$$\begin{aligned} a_1 + tb_1 &= c_1 + sd_1 \\ a_2 + tb_2 &= c_2 + sd_2 \\ a_3 + tb_3 &= c_3 + sd_3 \end{aligned}$$

and solve the system of equations for t and s. The number of solutions to the system determines the number of points the lines have in common.

Finding Points of Intersection

The line L_1 intersects the line L_2 at point Q. Find the coordinates of point Q.

$$L_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} \text{ and } L_2 = \begin{pmatrix} -1 \\ 2 \\ 15 \end{pmatrix} + s \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix}$$

Finding Points of Intersection

The line L_1 intersects the line L_2 at point Q. Find the coordinates of point Q.

$$L_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} \text{ and } L_2 = \begin{pmatrix} -1 \\ 2 \\ 15 \end{pmatrix} + s \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix}$$

$Q(4, -3, 5)$

The Angle Between Lines

Let θ be the angle between two lines in space. By definition, we always take this to be the smaller of the two angles formed by the lines. Hence $0^\circ \leq \theta \leq 90^\circ$.

Remember This!

To find the angle between two lines in space, we use the formula:

$$\cos\theta = \frac{|\vec{v} \cdot \vec{w}|}{|\vec{v}||\vec{w}|}$$

Exam Practice (14M.1.SL.TZ1.8)

The line L_1 passes through the points $A(2,1,4)$ and $B(1,1,5)$. Another line L_2 has equation $\vec{r} = \begin{pmatrix} 4 \\ 7 \\ -4 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. The lines L_1 and L_2 intersect at point P.

1. Show that $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.
2. Hence, write down a direction vector for L_1.
3. Hence, write down a vector equation for L_1.
4. Find the coordinates of P.
5. Write down a direction vector for L_2.
6. Hence, find the angle between L_1 and L_2

Exam Practice (14M.1.SL.TZ1.8)

The line L_1 passes through the points $A(2,1,4)$ and $B(1,1,5)$. Another line L_2 has equation $\vec{r} = \begin{pmatrix} 4 \\ 7 \\ -4 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. The lines L_1 and L_2 intersect at point P.

1. Show that $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. $\overrightarrow{AB} = \langle 1-2, 1-1, 5-4 \rangle = \langle -1, 0, 1 \rangle$
2. Hence, write down a direction vector for L_1. $\langle -1, 0, 1 \rangle$
3. Hence, write down a vector equation for L_1. $L_1 = \langle 2, 1, 4 \rangle + t\langle -1, 0, 1 \rangle$
4. Find the coordinates of P. $P(4, 1, 2)$
5. Write down a direction vector for L_2. $\langle 0, -1, 1 \rangle$
6. Hence, find the angle between L_1 and L_2 $\theta = 60^\circ$

Kinematics

Memorize This!

The position of an object moving along a straight path is given by the vector equation:

$$\vec{r} = \vec{p} + t\vec{v}$$

- $\vec{r}$ is the position of the object at time t
- $\vec{p}$ is the initial position of the object (at time $t = 0$)
- $\vec{v}$ is the velocity vector, where $|\vec{v}|$ is the speed of the object

Kinematics

Two cars are moving on straight roads according to the equations

$$\vec{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \end{pmatrix} \text{ and } \vec{r} = \begin{pmatrix} 1 \\ 13 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

1. Will the cars collide? Explain.
2. Find the minimum distance between the two cars.

Kinematics

Two cars are moving on straight roads according to the equations

$$\vec{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \end{pmatrix} \text{ and } \vec{r} = \begin{pmatrix} 1 \\ 13 \end{pmatrix} + t \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

1. Will the cars collide? Explain. No, although their paths intersect at $(5, 10)$, they reach this point at different times ($t = 2$ and $t = 1$, respectively)
2. Find the minimum distance between the two cars. At $t = 1.43$, their distance is a minimum of 6.22.

Equations of Lines and Planes

Additional Practice with Equations of Lines and Planes

Haese Mathematics SL (3rd Edition)

Chapter 25A #43, 65, 91

Chapter 25B #11, 34, 49, 53

Topic 5: Statistics and Probability

Topic 5: Statistics and Probability

- 5.1 One variable statistics
- 5.2 Two variable statistics
- 5.3 Elementary probability
- 5.4 Probability distributions

One-Variable Statistics

In this course (and on the IB exam), we only work with data sets where all the data is known. As a result, any given set of data represents the **population**.

When using your graphing caclulator:

- $\bar{x}$ is the **mean** of the data values
- σ_x is the **standard deviation** of the data values
- Σx is the sum of the data values
- Σx^2 is the sum of the squares of the data values
- n is the number of data values

We will **NOT** use s_x (the standard error)

Discrete and Continuous Data

- A **continuous** variable can take on any value in a given interval. Hence, continuous variables are measured. Examples include:
  - height
  - weight
  - volume
  - temperature
  - time
- A **discrete** variable takes on only distinct values. Hence, discrete variables are counted. Examples include:
  - number of students present
  - score on a test
  - retail price of an iPhone

Measures of Center

- The **mean** of the data is the average.
- The **median** of the data is a number such that half the data is below that number and half the data is above that number.
  - For an odd number of data values, the median is the middle value (when the data is arranged from least to greatest).
  - For an even number of data values, the median is the average of the two middle values.
- The **mode** of the data is the value with the highest frequency.

Measures of Center

x_i	f_i
1	4
2	2
3	4
5	2
6	5
7	3
9	1
13	2

Find the **mean**, **median**, and **mode** for the given set of data, where f_i represents the frequency of x_i.

Measures of Center

x_i	f_i
1	4
2	2
3	4
5	2
6	5
7	3
9	1
13	2

Find the **mean**, **median**, and **mode** for the given set of data, where f_i represents the frequency of x_i.

mean: 5.04

median: 5

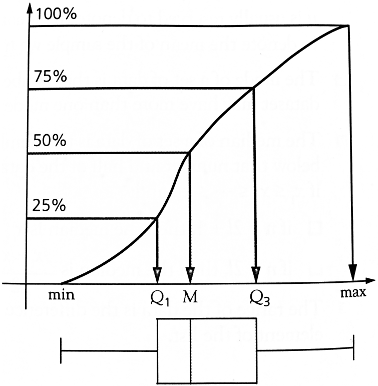
mode: 6

Percentiles

- The Pth percentile is the number such that P percent of the data is below that number.
- The **lower quartile** (first quartile) is denoted Q_1, and is the number such that 25% of the data lie below that number.
- The **upper quartile** (third quartile) is denoted Q_3, and is the number such that 75% of the data lie below that number.
- The **interquartile range** (IQR) is the difference between the upper and lower quartiles, $Q_3 - Q_1$.
- The **range** is the difference between the maximum and minimum values.

Cumulative Frequency

You may be asked to find percentiles using a cumulative frequency graph. The **five-number summary** can be used to construct a boxplot.



Five-Number Summary

x_i	f_i
1	4
2	2
3	4
5	2
6	5
7	3
9	1
13	2

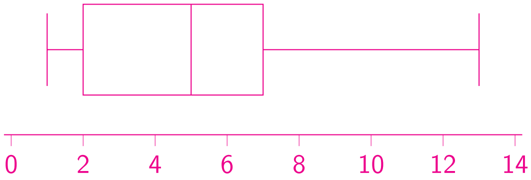
Find the five-number summary for the data in the frequency table, then use it to construct a boxplot.

Five-Number Summary

x_i	f_i
1	4
2	2
3	4
5	2
6	5
7	3
9	1
13	2

Find the five-number summary for the data in the frequency table, then use it to construct a boxplot.

min $= 1$, $Q_1 = 2$, Med $= 5$, $Q_3 = 7$, max $= 13$



Outliers

Memorize This!

An **outlier** is defined as any value more than 1.5 times the IQR below the lower quartile or 1.5 times the IQR above the upper quartile.

- x is a lower outlier if $x < Q_1 - 1.5 \times \text{IQR}$
- x is an upper outlier if $x > Q_3 + 1.5 \times \text{IQR}$

Measures of Spread

Memorize This!

- σ^2 is the **variance** of the data set
- σ is the **standard deviation** of the data set
- Hence, the square root of the variance gives the standard deviation

Measures of Center and Spread

Let x be a value from a data set with mean μ, median M, mode m, variance σ^2, and standard deviation σ.

If we multiply all of the x values in our dataset by a, then add b, then the new dataset has:

- mean: $a\mu + b$
- median: $aM + b$
- mode: $am + b$
- variance: $a^2\sigma^2$
- standard deviation: $|a|\sigma$

Remember, adding a constant simply shifts the measures of center, but has no effect on measures of spread. Multiplying by a constant affects measures of both center and spread.

Measures of Center and Spread

Summary statistics for a dataset x_i are given below. Find the new summary statistics for the dataset obtained when x_i is multiplied by 2 and increased by 5.

	x_i	$y_i = 2x_i + 5$
mean	$\mu_x = 7$	$\mu_y =$
median	$M_x = 6$	$M_y =$
mode	$m_x = 4$	$m_y =$
min	$\min_x = 1$	$\min_y =$
max	$\max_x = 10$	$\max_y =$
variance	$\sigma_x^2 = 9$	$\sigma_y^2 =$
st. dev.	$\sigma_x = 3$	$\sigma_y =$

Measures of Center and Spread

Summary statistics for a dataset x_i are given below. Find the new summary statistics for the dataset obtained when x_i is multiplied by 2 and increased by 5.

	x_i	$y_i = 2x_i + 5$
mean	$\mu_x = 7$	$\mu_y = 19$
median	$M_x = 6$	$M_y = 17$
mode	$m_x = 4$	$m_y = 13$
min	$\min_x = 1$	$\min_y = 7$
max	$\max_x = 10$	$\max_y = 25$
variance	$\sigma_x^2 = 9$	$\sigma_y^2 = 36$
st. dev.	$\sigma_x = 3$	$\sigma_y = 6$

One-Variable Statistics

Additional Practice with One-Variable Statistics

Haese Mathematics SL (3rd Edition)

Chapter 25A #14, 15, 31, 82, 92

Chapter 25B #13, 71

Bivariate Data

- Data for which two variables are measured is called **bivariate**.
- In a scatter diagram, the **independent variable**, x, is placed on the horizontal axis, and may be used to predict values of the **dependent variable**, y. Predicting x-values from y-values is not valid.
- The LSRL (Least Squares Regression Line) is a line of best fit which has the form $y = ax + b$.
- The regression line always passes through the mean point, $(\bar{x}, \bar{y})$.
- Making a prediction within the range of data values is called **interpolation**, and is considered valid.
- Making a prediction outside the range of data values is called **extrapolation**, and is not considered valid.

Correlation

- Pearson's correlation coefficient, r, measures the strength of the **linear** relationship between two variables.
- The correlation is a value between -1 and 1.
- The closer r is to -1 or 1, the stronger the correlation.
- The closer r is to 0, the weaker the correlation.
- Negative r-values indicate a **negative correlation**. Likewise, positive r-values indicate a **positive correlation**.
- **Correlation does not imply causation!**

Exam Practice (14M.2.SL.TZ1.3)

The following table shows the average weights (y kg) for given heights (x cm) in a population of men.

Heights (x cm)	165	170	175	180	185
Weights (y kg)	67.8	70.0	72.7	75.5	77.2

1. The relationship between the variables is modeled by the regression equation $y = ax + b$. Write down the value of a and b.
2. Hence, estimate the weight of a man whose height is 172 cm.
3. Write down the correlation coefficient.
4. State which **two** of the following describe the correlation between the variables: strong, zero, positive, negative, no correlation, weak.

Exam Practice (14M.2.SL.TZ1.3)

The following table shows the average weights (y kg) for given heights (x cm) in a population of men.

Heights (x cm)	165	170	175	180	185
Weights (y kg)	67.8	70.0	72.7	75.5	77.2

1. The relationship between the variables is modeled by the regression equation $y = ax + b$. Write down the value of a and b.
   $a = 0.486, b = -12.4$
2. Hence, estimate the weight of a man whose height is 172 cm.
   71.2 kg
3. Write down the correlation coefficient. $r = 0.997$
4. State which **two** of the following describe the correlation between the variables: strong, zero, positive, negative, no correlation, weak.
   strong, positive

Two-Variable Statistics

Additional Practice with Two-variable statistics

Haese Mathematics SL (3rd Edition)

Chapter 25A #38, 77

Chapter 25B #8, 9, 60, 76

Probability

- One repetition of an experiment is called a **trial**.
- The set of all possible outcomes in an experiment is called the **sample space**, and is denoted U (the universal set).
- The number of outcomes in the sample space is denoted $n(U)$.
- A set of outcomes from a sample space is called an **event**, and is denoted with a capital letter. For example, event A.
- A, $P(A)$, and $n(A)$ represent event A, the probability of event A, and the number of possible outcomes in event A, respectively.
- The **complement** of A is the set of outcomes that are not in A, and is denoted A'.

Probability Rules

Formulas

- $P(A) = \dfrac{n(A)}{n(U)}$
- $P(A) + P(A') = 1$

- Two events A and B are **mutually exclusive** if they have no outcomes in common. Hence A and A' are mutually exclusive.
- If A and B are mutually exclusive, then $A \cap B = \emptyset$, where $\emptyset$ represents the empty set, and $P(A \cap B) = 0$.

The Union of Two Events

Formulas

- The General Addition Rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

- If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

$A \cup B$ is called the **union** of events A and B. The union includes all of A along with all of B.

Independent Events

- Two events are independent if knowing that one occurs does not influence the probability that the other occurs.
- If A and B are independent, then A' and B' are also independent. Likewise, A and B', and B and A' are independent.
- The probability of B happening given that A has happened is denoted $P(B|A)$.
- If events A and B are independent, then $P(B|A) = P(B)$.

The Intersection of Two Events

Formulas

- Conditional Probability:

$$P(A \cap B) = P(A)P(B|A).$$

- If A and B are independent, then

$$P(A \cap B) = P(A)P(B).$$

$A \cap B$ is called the **intersection** of events A and B. The intersection includes only what A and B have in common.

Finding Probabilities

Methods for finding probabilities:

1. Counting principles
   - The number of ways to choose k items from n possible ojects is ${}^nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$
   - The number of ways to arrange n different objects in a row is $n!$
2. Venn diagrams
3. Tree diagrams
4. Tables of outcomes
5. Probability formulas

Tree Diagrams

P(A) =

- P(B | A) = → P(A and B) =
- P(B′ | A) = → P(A and B′) =

P(A′) =

- P(B | A′) = → P(A′ and B) =
- P(B′ | A′) = → P(A′ and B′) =

Exam Practice (14M.1.SL.TZ1.5)

Celeste wishes to hire a taxicab from a company which has a large number of taxicabs. The taxicabs are randomly assigned by the company.

- The probability that a taxicab is yellow is 0.4
- The probability that a taxicab is a Fiat is 0.3
- The probability that a taxicab is yellow or a Fiat is 0.6

Find the probability that the taxicab hired by Celeste is **not** a yellow Fiat.

Exam Practice (14M.1.SL.TZ1.5)

Celeste wishes to hire a taxicab from a company which has a large number of taxicabs. The taxicabs are randomly assigned by the company.

- The probability that a taxicab is yellow is 0.4
- The probability that a taxicab is a Fiat is 0.3
- The probability that a taxicab is yellow or a Fiat is 0.6

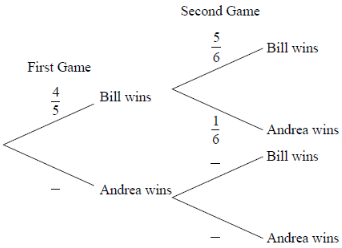
Find the probability that the taxicab hired by Celeste is **not** a yellow Fiat.

$P((Y \cap F)') = 0.9$

Exam Practice (14M.1.SL.TZ1.9)

Bill and Andrea play two games of tennis. The probability that Bill wins the first game is $\frac{4}{5}$. If Bill wins the first game, the probability that he wins the second game is $\frac{5}{6}$. If Bill loses the first game, the probability that he wins the second game is $\frac{2}{3}$.

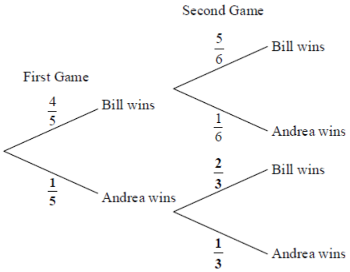
1. Copy and complete the following tree diagram.



Exam Practice (14M.1.SL.TZ1.9)

Bill and Andrea play two games of tennis. The probability that Bill wins the first game is $\frac{4}{5}$. If Bill wins the first game, the probability that he wins the second game is $\frac{5}{6}$. If Bill loses the first game, the probability that he wins the second game is $\frac{2}{3}$.

1. Copy and complete the following tree diagram.



Exam Practice (14M.1.SL.TZ1.9)

Bill and Andrea play two games of tennis. The probability that Bill wins the first game is $\frac{4}{5}$. If Bill wins the first game, the probability that he wins the second game is $\frac{5}{6}$. If Bill loses the first game, the probability that he wins the second game is $\frac{2}{3}$.

1.
2. Find the probability that bill wins the first game and Andrea wins the second game.
3. Find the probability that bill wins at least one game.
4. Given that Bill wins at least one game, find the probability that he wins both games.

Exam Practice (14M.1.SL.TZ1.9)

Bill and Andrea play two games of tennis. The probability that Bill wins the first game is $\frac{4}{5}$. If Bill wins the first game, the probability that he wins the second game is $\frac{5}{6}$. If Bill loses the first game, the probability that he wins the second game is $\frac{2}{3}$.

1.
2. Find the probability that bill wins the first game and Andrea wins the second game. $\frac{4}{30}$
3. Find the probability that bill wins at least one game. $\frac{28}{30}$
4. Given that Bill wins at least one game, find the probability that he wins both games. $\frac{20}{28}$

Additional Practice with Elementary Probability

Haese Mathematics SL (3rd Edition)

Chapter 25A #19, 48, 49, 67, 70, 73, 80, 84

Chapter 25B #14, 25, 29, 32, 56, 61

Random Variables

- A **discrete** random variable X has a countable number of possible values. For example:
  - the sum of two dice
  - the number of heads when flipping a coin four times
  - the number of cars in a parking lot.
- A **continuous** random variable takes on an infinite number of possible values on a given interval. For example:
  - the height of a randomly selected adult
  - the high temperature on a randomly selected day
  - the amount of time it takes to solve a puzzle.

Probability Distributions

- The **probability distribution** of a random variable describes the possible values of X, and the probabilities assigned to those values.
- Each probability is a value between 0 and 1.
- The sum of these probabilities must equal 1.

For example, let X represent the sum of two dice. Then the **probability distribution** of X is as follows:

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Expected Value

Formulas

The expected value of a discrete random variable X:

$$E(X) = \mu = \sum_x xP(X = x)$$

To compute the expected value of a discrete random variable X, find:

$$\sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$$

Expected Value

For the probability distribution

X	-5	0	5	20
$P(X = x_i)$	0.4	0.3	0.2	k

1. Find the value of k.
2. Find the expected value of X.

Expected Value

For the probability distribution

X	-5	0	5	20
$P(X = x_i)$	0.4	0.3	0.2	k

1. Find the value of k. $k = 0.1$
2. Find the expected value of X. $E(X) = 1$

Expected Value

A player has to put in 10 points to play a game. The probability of winning 15 points is 0.2 and the probability of winning W points in 0.001. In any other case, the player does not win any points. Find the value of W so that the game is fair.

X (winnings)	-10	5	
$P(X = x_i)$		0.2	0.001

Expected Value

A player has to put in 10 points to play a game. The probability of winning 15 points is 0.2 and the probability of winning W points in 0.001. In any other case, the player does not win any points. Find the value of W so that the game is fair.

X (winnings)	-10	5	
$P(X = x_i)$		0.2	0.001

$W = 7000$

Binomial Distribution

Let $X =$ the number of successes in n observations. Then $X \sim B(n, p)$ is called a **binomial** random variable if:

1. There are only two outcomes: **success** or **failure**
2. The probability of success p is **constant** for each observation
3. The n observations are **independent**

   Note that the independence assumption is violated whenever we sample without replacement, but is overridden by the 10% condition. As long as we don't sample more than 10% of the population, the probabilities don't change enough to matter.

4. There is a **fixed** number n of observations

Binomial Distribution

Formulas

- $X \sim B(n, p) \Rightarrow P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$, $r = 0, 1, \ldots, n$
- $E(X) = np$
- $\mathrm{Var}(X) = np(1-p)$

Binomial Distribution

- The **probability distribution function** (or PDF) assigns a probability to each value of X.

  So, to find $P(X = x)$ for a binomial random variable $B(n, p)$, we use `binompdf(n,p,x)` in the calculator.

- The **cumulative distribution function** (or CDF) calculates the sum of the probabilities up to X.

  So, to find $P(X \leq x)$ for a binomial random variable $B(n, p)$, we use `binomcdf(n,p,x)` in the calculator.

Binomial Distribution

Suppose each child born to Jay and Kay has probability 0.25 of having blood type O, and Jay and Kay have 5 children.

Let $X =$ the number of children with type O blood

1. Is X a binomial random variable? Explain.
2. Find the probability that exactly 2 of them have type O blood.
3. Find the probability that at least 2 of them have type O blood.

Binomial Distribution

Suppose each child born to Jay and Kay has probability 0.25 of having blood type O, and Jay and Kay have 5 children.

Let $X =$ the number of children with type O blood

1. Is X a binomial random variable? Explain. Yes, either a child has type O blood or doesn't, the probability of having type O is the same for each child, one child's blood type has no influence on the other children's blood type, and there are exactly five children.
2. Find the probability that exactly 2 of them have type O blood. $P(X = 2) = 0.264$
3. Find the probability that at least 2 of them have type O blood. $P(X \geq 2) = 1 - P(X \leq 1) = 0.367$

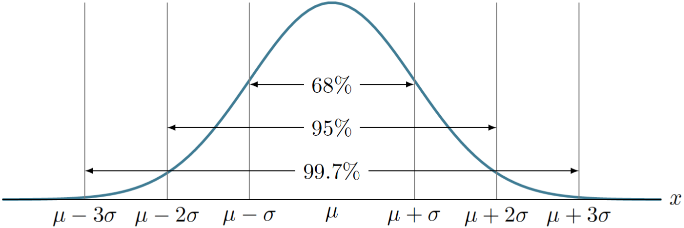
Continuous Random Variables

- A discrete random variable X has a countable number of possible values, but a **continuous** random variable X takes on all values in a given interval of numbers.
- The probability distribution of a continuous random variable is shown by a **density curve**. The total area under a density curve (no matter what shape it has) is 1.
- The probability that a continuous random variable falls between two x-values is the area under the probability density function between those two values. Therefore, the probability that a continuous random variable X is exactly equal to a number is 0.
- Hence, $P(a < X < b) = P(a \leq X \leq b)$ if X is a continuous random variable. Note: this is not true if X is a discrete random variable.

Normal Distribution

The **Normal** distribution is a symmetric, bell-shaped density curve centered at mean μ with standard deviation σ.

$$X \sim N(\mu, \sigma^2)$$



Normal Distribution

The probability that a random normal variable falls between two values can be found using your graphing calculator:

$$P(a < x < b) = \texttt{normalcdf(lower, upper, mean, SD)}$$

The average height of people in a town is 170 cm with standard deviation 10 cm. What is the probability that a randomly selected resident:

1. is less than 165 cm tall?
2. is between 180 cm and 190 cm tall?
3. is over 176 cm tall?

Normal Distribution

The probability that a random normal variable falls between two values can be found using your graphing calculator:

$$P(a < x < b) = \texttt{normalcdf(lower, upper, mean, SD)}$$

The average height of people in a town is 170 cm with standard deviation 10 cm. What is the probability that a randomly selected resident:

1. is less than 165 cm tall? 0.309
2. is between 180 cm and 190 cm tall? 0.136
3. is over 176 cm tall? 0.274

z-Scores

Formulas

The standardized normal variable, z, is given by:

$$z = \frac{x - \mu}{\sigma}$$

- The standard Normal distribution always has mean 0 and standard deviation 1.
- $P(a < x < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$

Inverse Normal Distribution

Remember This!

For a given value of probability p the inverse normal distribution gives the value of x such that

$$P(X \leq x) = p$$

This can be found using your graphing calculator:

$$X = \texttt{InvNorm(p,mean,SD)}$$

where p is the probability to the left of X.

Inverse Normal Distribution

1. The size of men's feet is thought to be normally distributed with mean 22 cm and variance 25 cm^2. A show manufacturer wants only 5% of men to be unable to find shoes large enough for them. How big should their largest shoe be?

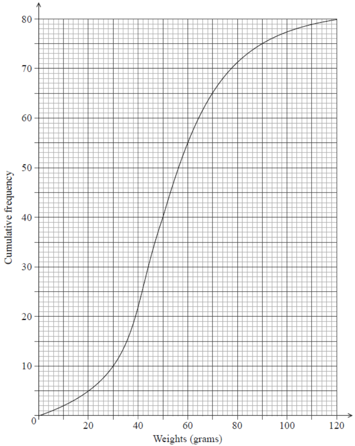
2. The masses of gerbils are thought to be normally distributed. If 30% of gerbils have a mass of more than 65 g and 20% have a mass of less than 40g, estimate the mean and the variance of the mass of a gerbil.

Inverse Normal Distribution

1. The size of men's feet is thought to be normally distributed with mean 22 cm and variance 25 cm^2. A show manufacturer wants only 5% of men to be unable to find shoes large enough for them. How big should their largest shoe be? 30.2 cm

2. The masses of gerbils are thought to be normally distributed. If 30% of gerbils have a mass of more than 65 g and 20% have a mass of less than 40g, estimate the mean and the variance of the mass of a gerbil. $\mu = 55.4, \sigma = 18.3$

Exam Practice (14M.2.SL.TZ1.8)

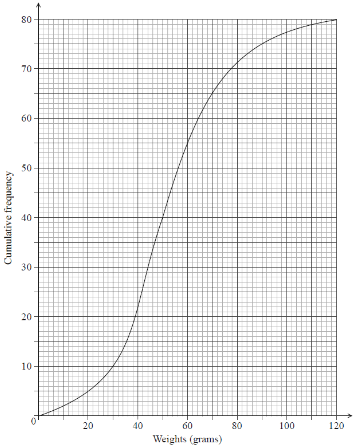


The weights of 80 rats are shown in the cumulative frequency diagram.

1. Write down the median weight of the rats.
2. Find the percentage of rats that weigh 70 grams or less.
3. Find the values of p and q.

Weights w grams	Frequency
$0 \leq w \leq 30$	p
$30 < w \leq 60$	45
$60 < w \leq 90$	q
$90 < w \leq 120$	5

Exam Practice (14M.2.SL.TZ1.8)



The weights of 80 rats are shown in the cumulative frequency diagram.

1. Write down the median weight of the rats. 50
2. Find the percentage of rats that weigh 70 grams or less. 81.25%
3. Find the values of p and q.
   $p = 10, q = 20$

Weights w grams	Frequency
$0 \leq w \leq 30$	p
$30 < w \leq 60$	45
$60 < w \leq 90$	q
$90 < w \leq 120$	5

Exam Practice (14M.2.SL.TZ1.8)

Weights w grams	$0 \leq w \leq 30$	$30 < w \leq 60$	$60 < w \leq 90$	$90 < w \leq 120$
Frequency	p	45	q	5

1. Use the values from the table to estimate the mean and standard deviation of the weights.

2. Assume that the weights of these rats are normally distributed with the mean and standard deviation estimated in the previous question. Find the percentage of rats that weigh 70 grams or less.

3. A sample of five rats is chosen at random. Find the probability that at most three rats weight 70 grams or less.

Exam Practice (14M.2.SL.TZ1.8)

Weights w grams	$0 \leq w \leq 30$	$30 < w \leq 60$	$60 < w \leq 90$	$90 < w \leq 120$
Frequency	p	45	q	5

1. Use the values from the table to estimate the mean and standard deviation of the weights. $\bar{x} = 52.5, \sigma = 22.5$
2. Assume that the weights of these rats are normally distributed with the mean and standard deviation estimated in the previous question. Find the percentage of rats that weigh 70 grams or less. 78.2%
3. A sample of five rats is chosen at random. Find the probability that at most three rats weight 70 grams or less. 0.301

Probability Distributions

Additional Practice with Probability Distributions

Haese Mathematics SL (3rd Edition)

Chapter 25A #27, 30, 47, 74, 75

Chapter 25B #15, 17, 19, 26, 27, 28, 30, 37, 38, 43, 45, 47, 52, 64, 65, 74

Topic 6: Calculus

- 6.1 Limit, convergence, continuity
- 6.2 Differential calculus
- 6.3 Integral calculus
- 6.4 Kinematics

Limit of a Function

Limit of a function

The number L is the limit of the function f at $x = a$ if the value of $f(x)$ gets arbitrarily close to L as x approaches a. We write:

$$\lim_{x \to a} f(x) = L.$$

1. $\lim_{x \to 0^+} \ln x =$
2. $\lim_{x \to \infty} \frac{3x-7}{4x+1} =$
3. $\lim_{x \to 3} \frac{1}{(x-3)^2} =$
4. $\lim_{x \to \infty} e^x =$

Limit of a Function

Limit of a function

The number L is the limit of the function f at $x = a$ if the value of $f(x)$ gets arbitrarily close to L as x approaches a. We write:

$$\lim_{x \to a} f(x) = L.$$

1. $\lim_{x \to 0^+} \ln x = -\infty$
2. $\lim_{x \to \infty} \frac{3x-7}{4x+1} = \frac{3}{4}$
3. $\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$
4. $\lim_{x \to \infty} e^x = \infty$

Derivative of $f(x)$

Formulas

- $y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \left(\frac{f(x+h)-f(x)}{h} \right)$

The derivative of a function f, at a point $x = a$, can also be described as:

- the slope (**gradient**) of the function
- the slope of the line tangent to the graph of f at $x = a$
- the rate of change of the function

First Principles

Find the derivative of $f(x) = \sqrt{x}$ from **first principles**.

First Principles

Find the derivative of $f(x) = \sqrt{x}$ from **first principles**.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \\
&= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h \to 0} \left(\frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h \to 0} \left(\frac{1}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Tangent and Normal Lines

Remember This!

- The equation of the **tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$ is

$$y = f'(a)(x - a) + f(a)$$

- The equation of the **normal line** to the graph of $y = f(x)$ at the point $(a, f(a))$ is

$$y = -\frac{1}{f'(a)}(x - a) + f(a)$$

Exam Practice (14M.2.SL.TZ1.7)

Let $f(x) = \frac{g(x)}{h(x)}$, where $g(2) = 18, h(2) = 6, g'(2) = 5$, and $h'(2) = 2$.

Find the equation of the normal to the graph of f at $x = 2$.

Exam Practice (14M.2.SL.TZ1.7)

Let $f(x) = \frac{g(x)}{h(x)}$, where $g(2) = 18, h(2) = 6, g'(2) = 5$, and $h'(2) = 2$.

Find the equation of the normal to the graph of f at $x = 2$.

$y - 3 = 6(x - 2)$

Derivative of $f(x)$

Formulas

- $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
- $f(x) = \sin x \Rightarrow f'(x) = \cos x$
- $f(x) = \cos x \Rightarrow f'(x) = -\sin x$
- $f(x) = \tan x \Rightarrow f'(x) = \frac{1}{\cos^2 x}$
- $f(x) = e^x \Rightarrow f'(x) = e^x$
- $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

Derivative Rules

Formulas

- Chain Rule: $y = g(u), u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
- Product Rule: $y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
- Quotient Rule: $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

The Chain Rule

Use the chain rule to find y'.

1. $y = \sin^3(x)$
2. $y = \sqrt{x^4 - 1}$
3. $y = \ln(3x + 1)$
4. $y = \cos(\ln(x^2))$
5. $y = e^{\tan x}$

The Chain Rule

Use the chain rule to find y'.

1. $y = \sin^3(x)$ $\quad y' = 3\sin^2 x \cos x$
2. $y = \sqrt{x^4 - 1}$ $\quad y' = \frac{2x^3}{\sqrt{x^4-1}}$
3. $y = \ln(3x+1)$ $\quad y' = \frac{3}{3x+1}$
4. $y = \cos(\ln(x^2))$ $\quad y' = -\sin(\ln(x^2)) \cdot \frac{2x}{x^2} = \frac{-2\sin(\ln(x^2))}{x}$
5. $y = e^{\tan x}$ $\quad y' = e^{\tan x} \cdot \sec^2 x$

Exam Practice (14M.2.SL.TZ1.5)

The population of deer in an enclosed game reserve is modeled by the function $P(t) = 210\sin(0.5t - 2.6) + 990$, where t is in months, and $t = 1$ corresponds to 1 January 2014.

1. Find the number of deer in ther reserve on 1 May 2014.
2. Find the rate of change of the deer population on 1 May 2014.
3. Interpret the answer to part (i) with the reference to the deer population size on 1 May 2014.

Exam Practice (14M.2.SL.TZ1.5)

The population of deer in an enclosed game reserve is modeled by the function $P(t) = 210\sin(0.5t - 2.6) + 990$, where t is in months, and $t = 1$ corresponds to 1 January 2014.

1. Find the number of deer in ther reserve on 1 May 2014. 969 deer
2. Find the rate of change of the deer population on 1 May 2014. 104 deer per month
3. Interpret the answer to part (i) with the reference to the deer population size on 1 May 2014. the deer population is increasing

Exam Practice (14M.1.SL.TZ1.7)

Let $f(x) = px^3 + px^2 + qx$.

1. Find $f'(x)$.
2. Given that $f'(x) \geq 0$, show that $p^2 \leq 3pq$.

Exam Practice (14M.1.SL.TZ1.7)

Let $f(x) = px^3 + px^2 + qx$.

1. Find $f'(x)$. $f'(x) = 3px^2 + 2px + q$
2. Given that $f'(x) \geq 0$, show that $p^2 \leq 3pq$.
   $\Delta = (2p)^2 - 4(3p)(q) = 4p^2 - 12pq \leq 0 \Rightarrow 4p^2 \leq 12pq \Rightarrow p^2 \leq 3pq$

Derivative Graphs

- If $f'(x) > 0$ for all $x \in [a, b]$, then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for all $x \in [a, b]$, then f is decreasing on $[a, b]$.
- f has a **stationary point** at $x = a$ if $f'(a) = 0$.
- f has a local minimum at $x = a$ if f' changes from negative to positive at $x = a$.
- f has a local maximum at $x = a$ if f' changes from positive to negative at $x = a$.

Exam Practice (14M.2.SL.TZ1.10)

Let $f(x) = \dfrac{3x}{x-q}$, where $x \neq q$.

1. Write down the equations of the vertical and horizontal asymptotes of the graph of f.
2. The vertical and horizontal asymptotes to the graph of f intersect at the point $Q(1,3)$. Find the value of q.
3. The point $P(x,y)$ lies on the graph of f. Show that $PQ = \sqrt{(x-1)^2 + (\frac{3}{x-1})^2}$.
4. Hence find the coordinates of the points on the graph of f that are closest to $(1,3)$.

Exam Practice (14M.2.SL.TZ1.10)

Let $f(x) = \dfrac{3x}{x-q}$, where $x \neq q$.

1. Write down the equations of the vertical and horizontal asymptotes of the graph of f. $x = 1, y = 3$
2. The vertical and horizontal asymptotes to the graph of f intersect at the point $Q(1, 3)$. Find the value of q. $q = 1$
3. The point $P(x, y)$ lies on the graph of f. Show that $PQ = \sqrt{(x-1)^2 + (\frac{3}{x-1})^2}$.
4. Hence find the coordinates of the points on the graph of f that are closest to $(1, 3)$. $(-0.732, 1.27), (2.73, 4.73)$

Derivative Graphs

- If $f''(x) > 0$ for all $x \in [a, b]$, then f is concave up on $[a, b]$.
- If $f''(x) < 0$ for all $x \in [a, b]$, then f is concave down on $[a, b]$.
- f has a point of inflection at $x = a$ if f'' changes from sign at $x = a$.
- If f is concave up on $[a, b]$, then the graph is above any tangent line drawn in that interval, and is below any secant line drawn between two points in that interval.
- If f is concave down on $[a, b]$, then the graph is below any tangent line drawn in that interval, and is above any secant line drawn between two points in that interval.

Derivative Graphs

- If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at $x = a$.
- If $f'(a) = 0$ and $f''(a) < 0$, then f has a local maximum at $x = a$.
- f has a point of inflection where f' has a local max or min.

Differential Calculus

Additional Practice with Differential Calculus

Haese Mathematics SL (3rd Edition)

Chapter 25A #17, 18, 20, 21, 45, 53, 54, 55, 66, 72, 90

Chapter 25B #2, 6, 18, 21, 46, 50, 68, 70, 75

Integrals

Formulas

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq 1$
- $\int \frac{1}{x} \, dx = \ln x + C, x > 0$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int e^x \, dx = e^x + C$

U-Substitution

Use an appropriate substitution to evaluate the following:

1. $\int 3x^2(x^3+1)^5\,dx$
2. $\int x\cos x^2\,dx$
3. $\int_1^2 \frac{6x+3}{(x^2+x)^3}\,dx$

U-Substitution

Use an appropriate substitution to evaluate the following:

1. $\int 3x^2(x^3+1)^5\,dx = \frac{(x^3+1)^6}{6} + C$
2. $\int x\cos x^2\,dx = \frac{1}{2}\sin x^2 + C$
3. $\int_1^2 \frac{6x+3}{(x^2+x)^3}\,dx = \left[\frac{-3}{2(x^2+x)^2}\right]_1^2 = \frac{1}{3}$

Exam Practice (14M.1.SL.TZ1.6)

Let $\int_{\pi}^{a} \cos 2x \, dx = \frac{1}{2}$, where $\pi < a < 2\pi$. Find the value of a.

Exam Practice (14M.1.SL.TZ1.6)

Let $\int_{\pi}^{a} \cos 2x \, dx = \frac{1}{2}$, where $\pi < a < 2\pi$. Find the value of a.

$a = \frac{5\pi}{4}$

Properties of Integrals and the FTC

Remember This!

- $\int_a^a f(x)\,dx = 0$
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$
- $\int_a^b f'(x)\,dx = f(b) - f(a)$
- $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$

Integrals

Formulas

- Area under a curve between $x = a$ and $x = b$

$$A = \int_a^b y \, dx$$

- Volume of revolution about the x-axis from $x = a$ to $x = b$

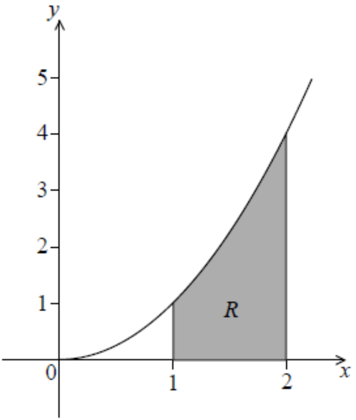
$$V = \int_a^b \pi y^2 \, dx$$

Exam Practice (14M.1.SL.TZ1.3)

Let $f(x) = x^2$.

1. Find $\int_1^2 (f(x))^2 \, dx$.

2. The following diagram shows part of the graph of f. The shaded region R is enclosed by the graph of f, the x-axis and the line $x = 1$ and $x = 2$. Find the volume of the solid formed when R is revolved 360° about the x-axis.

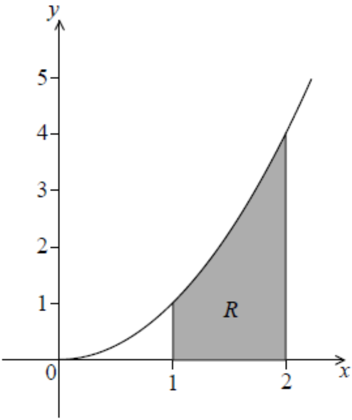


Exam Practice (14M.1.SL.TZ1.3)

Let $f(x) = x^2$.

1. Find $\int_1^2 (f(x))^2\, dx$. $\frac{31}{5}$

2. The following diagram shows part of the graph of f. The shaded region R is enclosed by the graph of f, the x-axis and the line $x = 1$ and $x = 2$. Find the volume of the solid formed when R is revolved 360° about the x-axis. $\frac{31\pi}{5}$



Integral Calculus

Additional Practice with Integral Calculus

Haese Mathematics SL (3rd Edition)

Chapter 25A #25, 26, 41, 46d, 56, 57, 59b, 69, 88, 89

Chapter 25B #20, 23, 24, 35, 36, 40, 41, 51, 55

Kinematics

- The function $t \mapsto s(t)$ represents the position of an object at time t.
- The function $t \mapsto v(t)$ represents the velocity of an object at time t.
  - Positive velocity indicates movement in the positive direction.
  - Negative velocity indicates movement in the negative direction.
  - An object's velocity when it is changing direction is zero.
  - Speed is the absolute value of the velocity.
- The function $t \mapsto a(t)$ represents the acceleration of an object at time t.
  - If the velocity and acceleration of an object have the same sign, then the object is speeding up.
  - If the velocity and acceleration of an object have opposite signs, then the object is slowing down.

Kinematics

Formulas

Total distance travelled from t_1 to t_2:

$$\text{distance} = \int_{t^1}^{t_2} |v(t)|\, dt$$

Exam Practice (14M.2.SL.TZ1.6)

Ramiro and Lautaro are travelling from Buenos Aires to El Moro. Ramiro travels in a vehicle whose velocity in $ms^{-}1$ is given by $V_R = 40 - t^2$, where t is in seconds. Lautaro travels in a vehicle whose displacement from Buenos Aires in metres is given by $S_L = 2t^2 + 60$. When $t = 0$, both vehicles are at the same point. Find Ramiro's displacement from Buenos Aires when $t = 10$.

Exam Practice (14M.2.SL.TZ1.6)

Ramiro and Lautaro are travelling from Buenos Aires to El Moro. Ramiro travels in a vehicle whose velocity in $ms^{-}1$ is given by $V_R = 40 - t^2$, where t is in seconds. Lautaro travels in a vehicle whose displacement from Buenos Aires in metres is given by $S_L = 2t^2 + 60$. When $t = 0$, both vehicles are at the same point. Find Ramiro's displacement from Buenos Aires when $t = 10$.

127 m

Kinematics

Additional Practice with Kinematics

Haese Mathematics SL (3rd Edition)

Chapter 25A #22, 24, 33, 37, 50