FINANCIAL GROWTH

Suppose an amount u_1 is invested at a fixed rate for each compounding period. In this case the value of the investment after n periods is given by $u_{n+1} = u_1 \times r^n$ where r is the multiplier corresponding to the given rate of interest. In order to find n algebraically, we need to use **logarithms**.

Example 27 Self Tutor

Iryna has €5000 to invest in an account that pays 5.2% p.a. interest compounded annually. Find, using logarithms, how long it will take for her investment to reach €20 000.

$u_{n+1} = 20\,000$ after n years
$u_1 = 5000$
$r = 105.2\% = 1.052$

Now $u_{n+1} = u_1 \times r^n$
$\therefore \quad 20\,000 = 5000 \times (1.052)^n$
$\therefore \quad (1.052)^n = 4$
$\therefore \quad \log(1.052)^n = \log 4$
$\therefore \quad n \times \log 1.052 = \log 4$
$\therefore \quad n = \dfrac{\log 4}{\log 1.052} \approx 27.3$ years

Rounding up here, it will take about 28 years to reach €20 000.

EXERCISE 4H

1 The weight W_t of bacteria in a culture t hours after establishment is given by $W_t = 20 \times 2^{0.15t}$ grams. Find, using logarithms, the time for the weight of the culture to reach:

a 30 grams **b** 100 grams.

2 The mass M_t of bacteria in a culture t hours after establishment is given by $M_t = 25 \times e^{0.1t}$ grams. Show that the time required for the mass of the culture to reach 50 grams is $10 \ln 2$ hours.

3 A biologist is modelling an infestation of fire ants. He determines that the area affected by the ants is given by $A_n = 2000 \times e^{0.57n}$ hectares, where n is the number of weeks after the initial observation.

a Draw an accurate graph of A_n against n.

b Use your graph to estimate the time taken for the infested area to reach 10 000 ha.

c Check your answer to **b** using: **i** logarithms **ii** suitable technology.

4 A house is expected to increase in value at an average rate of 7.5% p.a. If the house is worth £160 000 now, when would you expect it to be worth £250 000?

5 Thabo has $10 000 to invest in an account that pays 4.8% p.a. compounded annually. How long will it take for his investment to grow to $15 000?

6 Dien invests $15 000 at 8.4% p.a. compounded *monthly*. He will withdraw his money when it reaches $25 000, at which time he plans to travel. The formula $u_{n+1} = u_1 \times r^n$ can be used to model the investment, where n is the time in months.

a Explain why $r = 1.007$. **b** After how many months will Dien withdraw the money?

7 The mass M_t of radioactive substance remaining after t years is given by $M_t = 1000 \times e^{-0.04t}$ grams. Find the time taken for the mass to:

a halve **b** reach 25 grams **c** reach 1% of its original value.

8 A man jumps from an aeroplane. His speed of descent is given by $V = 50(1 - e^{-0.2t})$ m s^{-1}, where t is the time in seconds. Show that it will take $5 \ln 5$ seconds for the man's speed to reach 40 m s^{-1}.

9 Answer the **Opening Problem** on page **110**.

10 The temperature of a liquid t minutes after it is placed in a refrigerator, is given by $T = 4 + 96 \times e^{-0.03t}$ °C. Find the time required for the temperature to reach:

a 25°C **b** 5°C.

11 The weight of radioactive substance remaining after t years is given by $W = 1000 \times 2^{-0.04t}$ grams.

a Sketch the graph of W against t.

b Write a function for t in terms of W.

c Hence find the time required for the weight to reach:

i 20 grams **ii** 0.001 grams.

12 The weight of radioactive uranium remaining after t years is given by the formula $W(t) = W_0 \times 2^{-0.0002t}$ grams, $t \geqslant 0$. Find the time required for the weight to fall to:

a 25% of its original value **b** 0.1% of its original value.

13 The current I flowing in a transistor radio t seconds after it is switched off is given by $I = I_0 \times 2^{-0.02t}$ amps. Show that it takes $\frac{50}{\log 2}$ seconds for the current to drop to 10% of its original value.

14 A parachutist jumps from the basket of a stationary hot air balloon. His speed of descent is given by $V = 60(1 - 2^{-0.2t})$ m s^{-1} where t is the time in seconds. Find the time taken for his speed to reach 50 m s^{-1}.

REVIEW SET 4A NON-CALCULATOR

1 Find the following, showing all working.

a $\log_4 64$ **b** $\log_2 256$ **c** $\log_2(0.25)$ **d** $\log_{25} 5$

e $\log_8 1$ **f** $\log_{81} 3$ **g** $\log_9(0.\overline{1})$ **h** $\log_k \sqrt{k}$

2 Find:

a $\log \sqrt{10}$ **b** $\log \frac{1}{\sqrt[3]{10}}$ **c** $\log(10^a \times 10^{b+1})$

3 Simplify:

a $4 \ln 2 + 2 \ln 3$ **b** $\frac{1}{2} \ln 9 - \ln 2$ **c** $2 \ln 5 - 1$ **d** $\frac{1}{4} \ln 81$

4 Find:

a $\ln(e\sqrt{e})$ **b** $\ln\left(\frac{1}{e^3}\right)$ **c** $\ln(e^{2x})$ **d** $\ln\left(\frac{e}{e^x}\right)$

5 Write as a single logarithm:

a $\log 16 + 2 \log 3$ **b** $\log_2 16 - 2 \log_2 3$ **c** $2 + \log_4 5$

6 Write as logarithmic equations:

a $P = 3 \times b^x$ **b** $m = \frac{n^3}{p^2}$

7 Show that $\log_3 7 \times 2\log_7 x = 2\log_3 x$.

8 Write the following equations without logarithms:

a $\log T = 2\log x - \log y$ **b** $\log_2 K = \log_2 n + \frac{1}{2}\log_2 t$

9 Write in the form $a \ln k$ where a and k are positive whole numbers and k is prime:

a $\ln 32$ **b** $\ln 125$ **c** $\ln 729$

10 Copy and complete:

Function	$y = \log_2 x$	$y = \ln(x+5)$
Domain		
Range		

11 If $A = \log_5 2$ and $B = \log_5 3$, write in terms of A and B:

a $\log_5 36$ **b** $\log_5 54$ **c** $\log_5(8\sqrt{3})$ **d** $\log_5(20.25)$ **e** $\log_5(0.\overline{8})$

12 Solve for x:

a $3e^x - 5 = -2e^{-x}$ **b** $2\ln x - 3\ln\left(\frac{1}{x}\right) = 10$

REVIEW SET 4B CALCULATOR

1 Write in the form 10^x giving x correct to 4 decimal places:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

2 Find x if:

a $\log_2 x = -3$ **b** $\log_5 x \approx 2.743$ **c** $\log_3 x \approx -3.145$

3 Write the following equations without logarithms:

a $\log_2 k \approx 1.699 + x$ **b** $\log_a Q = 3\log_a P + \log_a R$

c $\log A \approx 5\log B - 2.602$

4 Solve for x, giving exact answers:

a $5^x = 7$ **b** $20 \times 2^{2x+1} = 640$

5 The weight of a radioactive isotope after t years is given by $W_t = 2500 \times 3^{-\frac{t}{3000}}$ grams.

a Find the initial weight of the isotope.

b Find the time taken for the isotope to reduce to 30% of its original weight.

c Find the percentage weight loss after 1500 years.

d Sketch the graph of W_t against t.

6 Show that the solution to $16^x - 5 \times 8^x = 0$ is $x = \log_2 5$.

7 Solve for x, giving exact answers:

a $\ln x = 5$ **b** $3\ln x + 2 = 0$ **c** $e^x = 400$

d $e^{2x+1} = 11$ **e** $25e^{\frac{x}{2}} = 750$

8 A population of seals is given by $P_t = P_0 2^{\frac{t}{3}}$ where t is the time in years, $t \geqslant 0$.

a Find the time required for the population to double in size.

b Find the percentage increase in population during the first 4 years.

9 Consider $g : x \mapsto 2e^x - 5$.

a Find the defining equation of g^{-1}.

b Sketch the graphs of g and g^{-1} on the same set of axes.

c State the domain and range of g and g^{-1}.

d State the asymptotes and intercepts of g and g^{-1}.

10 Consider $f(x) = e^x$ and $g(x) = \ln(x+4)$, $x > -4$. Find:

a $(f \circ g)(5)$ **b** $(g \circ f)(0)$

REVIEW SET 4C

1 Without using a calculator, find the base 10 logarithms of:

a $\sqrt{1000}$ **b** $\dfrac{10}{\sqrt[3]{10}}$ **c** $\dfrac{10^a}{10^{-b}}$

2 Simplify:

a $e^{4\ln x}$ **b** $\ln(e^5)$ **c** $\ln(\sqrt{e})$

d $10^{\log x + \log 3}$ **e** $\ln\left(\dfrac{1}{e^x}\right)$ **f** $\dfrac{\log x^2}{\log_3 9}$

3 Write in the form e^x, where x is correct to 4 decimal places:

a 20 **b** 3000 **c** 0.075

4 Solve for x:

a $\log x = 3$ **b** $\log_3(x+2) = 1.732$ **c** $\log_2\left(\dfrac{x}{10}\right) = -0.671$

5 Write as a single logarithm:

a $\ln 60 - \ln 20$ **b** $\ln 4 + \ln 1$ **c** $\ln 200 - \ln 8 + \ln 5$

6 Write as logarithmic equations:

a $M = ab^n$ **b** $T = \dfrac{5}{\sqrt{l}}$ **c** $G = \dfrac{a^2 b}{c}$

7 Solve for x:

a $3^x = 300$ **b** $30 \times 5^{1-x} = 0.15$ **c** $3^{x+2} = 2^{1-x}$

8 Solve exactly for x:

a $e^{2x} = 3e^x$ **b** $e^{2x} - 7e^x + 12 = 0$

9 Write the following equations without logarithms:

a $\ln P = 1.5 \ln Q + \ln T$ **b** $\ln M = 1.2 - 0.5 \ln N$

10 For the function $g : x \mapsto \log_3(x+2) - 2$:

a Find the domain and range.

b Find any asymptotes and axes intercepts for the graph of the function.

c Find the defining equation for g^{-1}. Explain how to verify your answer.

d Sketch the graphs of g, g^{-1}, and $y = x$ on the same axes.

11 The weight of a radioactive isotope remaining after t weeks is given by $W_t = 8000 \times e^{-\frac{t}{20}}$ grams. Find the time for the weight to:

a halve **b** reach 1000 g **c** reach 0.1% of its original value.