Calc AB/SL2 Unit 7

1. $\frac{d}{dx}\int_2^{3x}(2t+3)\,dt = (2(3x)+3)(3) = \boxed{18x+9}$

2. $\frac{d}{dx}\int_{-2}^{x^4} 3\sqrt{t}\,dt = 3\sqrt{x^4}\cdot 4x^3 = \boxed{12x^5}$

3. $\frac{d}{dx}\int_{2x}^{-1}(t^2+2t)\,dt = -((2x)^2+2(2x))(2) = \boxed{-8x^2-8x}$

4. $\frac{d}{dx}\int_{-5}^{\cos x} 2t^2\,dt = \boxed{-2\cos^2 x\sin x}$

5. $\frac{d}{dx}\int_2^{x^2+2x}(3t-2)\,dt = (3(x^2+2x)-2)(2x+2) = (3x^2+6x-2)(2x+2)$

$= 6x^3+12x^2-4x+6x^2+12x-4$

$= \boxed{6x^3+18x^2+8x-4}$

6. $\frac{d}{dx}\int_{\ln x}^{2}(e^t+t)\,dt = -(e^{\ln x}+\ln x)\frac{1}{x} = \boxed{-1-\frac{\ln x}{x}}$

7. $g'(x) = \frac{d}{dx}\int_0^x t^3e^t\,dt = x^3e^x$ $\qquad \boxed{g'(1)=e}$

$g''(x) = [x^3e^x]' = 3x^2e^x + x^3e^x$ $\qquad g''(1) = 3e+e = \boxed{4e}$

8. $h'(x) = \frac{d}{dx}\int_{x^2}^{2}\sqrt{1+t^4}\,dt = -(\sqrt{1+x^8})(2x) = \boxed{-2x\sqrt{1+x^8}}$

$h''(x) = [-2x\sqrt{1+x^8}]' = -2\sqrt{1+x^8} - 2x\left(\frac{8x^7}{2\sqrt{1+x^8}}\right)$

$h''(1) = -2\sqrt{2} - 2\left(\frac{8}{2\sqrt{2}}\right) = -2\sqrt{2} - \frac{8}{\sqrt{2}} = -2\sqrt{2} - 4\sqrt{2} = \boxed{-6\sqrt{2}}$

9. a) $F(0) = \int_{-6}^{0} f(t)\,dt = \boxed{-2\pi+2}$

b) $F(-\frac{1}{2}) = \int_{-6}^{-1} f(t)\,dt = \boxed{-2\pi+\frac{1}{2}}$

c) $F'(x) = 2f(2x)$

$F'(-2) = 2f(-4) = \boxed{-4}$

d) $F'(2.5) = 2f(5) = \boxed{4}$

e) $F''(x) = 4f'(2x)$

$F''(0) = 4f'(0) = \boxed{4}$

10. a) $G(3) = \int_{-2}^{3} f(t)\,dt = -3-(8-\pi) = \boxed{\pi - 11}$

b) $G(-4) = \int_{-2}^{-4} f(t)\,dt = \boxed{-6}$

c) $G'(x) = f(x) \qquad G'(-2) = f(-2) = \boxed{2}$

d) $G''(x) = f'(x) \qquad G''(-5) = f'(-5) = \boxed{2}$

11. $\int x^3(x^4+3)^3\,dx \qquad u = x^4+3$

$\frac{du}{dx} = 4x^3$

$x^3\,dx = \frac{1}{4}du$

$\frac{1}{4}\int u^3\,du = \frac{1}{4}\cdot\frac{u^4}{4} + C = \boxed{\frac{1}{16}(x^4+3)^4 + C}$

12. $\int x\sqrt[3]{1+2x^2}\,dx \qquad u = 1+2x^2$

$\frac{du}{dx} = 4x$

$x\,dx = \frac{1}{4}du$

$\frac{1}{4}\int \sqrt[3]{u}\,du = \frac{1}{4}\int u^{1/3}\,du = \frac{1}{4}\cdot\frac{3}{4}u^{\frac{4}{3}} + C = \boxed{\frac{3}{16}(1+2x^2)^{\frac{4}{3}} + C}$

13. $\int x^3 \sin x^4\,dx \qquad u = x^4$

$\frac{du}{dx} = 4x^3$

$x^3\,dx = \frac{1}{4}du$

$\frac{1}{4}\int \sin u\,du = \boxed{-\frac{1}{4}\cos x^4 + C}$

14. $\int \frac{x^3}{(1+x^4)^2}\,dx$ $\quad u = 1 + x^4$

$\frac{du}{dx} = 4x^3$

$x^3 dx = \frac{1}{4} du$

$\frac{1}{4}\int \frac{1}{u^2}du = \frac{1}{4}\int u^{-2} du = -\frac{1}{4u} + C = \boxed{-\frac{1}{4+4x^4} + C}$

15. $\int 5x\sqrt{1-x^2}\,dx$ $\quad u = 1 - x^2$

$\frac{du}{dx} = -2x$

$x\,dx = -\frac{1}{2}du$

$-\frac{5}{2}\int \sqrt{u}\,du = -\frac{5}{2}\int u^{\frac{1}{2}}du = -\frac{5}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \boxed{-\frac{5}{3}(1-x^2)^{\frac{3}{2}} + C}$

16. $\int u^2\sqrt{u^3+2}\,du$ $\quad w = u^3 + 2$

$\frac{dw}{du} = 3u^2$

$u^2 du = \frac{1}{3}dw$

$\frac{1}{3}\int \sqrt{w}\,dw = \frac{1}{3}\int w^{1/2}dw = \frac{1}{3}\cdot\frac{2}{3}w^{\frac{3}{2}} + C = \boxed{\frac{2}{9}(u^3+2)^{\frac{3}{2}} + C}$

17. $\int x\sqrt{2x-1}\,dx$ $\quad u = 2x - 1$ $\quad x = \frac{u+1}{2}$

$\frac{du}{dx} = 2$

$dx = \frac{1}{2}du$

$\frac{1}{2}\int x\sqrt{u}\,du = \frac{1}{2}\int \left(\frac{u+1}{2}\right)\sqrt{u}\,du = \frac{1}{4}\int (u\sqrt{u} + \sqrt{u})\,du$

$= \frac{1}{4}\int (u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du = \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right] + C$

$= \boxed{\frac{1}{10}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + C}$

18. $\int (x+1)\sqrt{2-x}\,dx$ $\quad u = 2-x \quad x = 2-u$

$\frac{du}{dx} = -1$

$dx = -du$

$$-\int (2-u+1)\sqrt{u}\,du = -\int (3-u)\sqrt{u}\,du = -\int (3\sqrt{u} - u\sqrt{u})\,du$$

$$= -\int (3u^{1/2} - u^{3/2})\,du = -\left[3 \cdot \tfrac{2}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right] + C$$

$$= \boxed{-2(2-x)^{3/2} + \tfrac{2}{5}(2-x)^{5/2} + C}$$

19. $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$ $\quad u = 2x-1 \quad x = \frac{u+1}{2}$

$\frac{du}{dx} = 2$

$dx = \frac{1}{2}du$

$$= \frac{1}{2}\int_\alpha^\beta \frac{u+1}{2\sqrt{u}}\,du = \frac{1}{4}\int_\alpha^\beta \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right)du = \frac{1}{4}\int_\alpha^\beta \left(u^{1/2} + \frac{2}{2\sqrt{u}}\right)du$$

$$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2\sqrt{u}\right]_\alpha^\beta = \frac{1}{4}\left[\frac{2}{3}(2x-1)^{3/2} + 2\sqrt{2x-1}\right]_1^5$$

$$= \frac{1}{4}\left[\left(\frac{2}{3}\cdot 27 + 2\cdot 3\right) - \left(\frac{2}{3} + 2\right)\right] = \boxed{\frac{16}{3}}$$

20. a) $\frac{dy}{dx} = y^2(6-2x)$ $\quad \left.\frac{dy}{dx}\right|_{(3,\frac{1}{4})} = \left(\frac{1}{4}\right)^2(6-2(3)) = 0$

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6-2x) + y^2(-2)$$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,\frac{1}{4})} = 2\left(\frac{1}{4}\right)(0)(6-6) + \left(\frac{1}{4}\right)^2(-2) = \boxed{-\frac{1}{8}}$$

b) $\frac{dy}{dx} = y^2(6-2x)$

$\int \frac{1}{y^2}\,dy = \int (6-2x)\,dx$

$-\frac{1}{y} = 6x - x^2 + C$

$\frac{1}{y} = x^2 - 6x + C$

$y = \frac{1}{x^2 - 6x + C}$

$\frac{1}{4} = \frac{1}{9 - 18 + C} \Rightarrow 4 = -9 + C \Rightarrow C = 13$

$\boxed{y = \frac{1}{x^2 - 6x + 13}}$

21. $\frac{dy}{dx} = \frac{3-x}{y}$

a) $\int y\,dy = \int (3-x)\,dx$

$\frac{y^2}{2} = 3x - \frac{x^2}{2} + C$

$y^2 = 6x - x^2 + C$

Tangent: $y + 2 = 0(x - x_1)$

$\left.\frac{dy}{dx}\right|_{(x_1,-2)} = 0$

$\frac{3 - x_1}{-2} = 0$

$\boxed{x_1 = 3}$

$\left.\frac{dy}{dx}\right|_{(3,-2)} = 0$

$\frac{d^2y}{dx^2} = \frac{-y - (3-x)\frac{dy}{dx}}{y^2}$

$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \frac{2-0}{4} = \frac{1}{2} > 0$

concave up

$\therefore$ f has a local minimum by the Second Derivative Test

b) $y^2 = 6x - x^2 + C$

$16 = 6(6) - 36 + C \Rightarrow C = 16$

$$\boxed{y = -\sqrt{6x - x^2 + 16}}$$

22. $\frac{dy}{dt} = ky$

a) $\int \frac{1}{y}\,dy = \int k\,dt$

$\ln|y| = kt + C$

$|y| = Ce^{kt}$

$y = Ce^{kt}$

$1{,}000{,}000 = Ce^{\cancel{0}}$

$y = 1{,}000{,}000e^{kt}$

$500{,}000 = 1{,}000{,}000e^{6k}$

$\frac{1}{2} = e^{6k}$

$\ln\frac{1}{2} = 6k \Rightarrow k = \frac{\ln\frac{1}{2}}{6} = -0.1155$

$$\boxed{y = 1{,}000{,}000e^{-0.1155t}}$$

b) $\left.\frac{dy}{dt}\right|_{y=600{,}000} = -0.1155(600{,}000) = \boxed{-69{,}300 \text{ gallons/yr}}$

c) $1{,}000{,}000e^{-0.1155t} \le 50{,}000$ $\boxed{t \ge 25.9 \text{ years}}$

23. $\frac{dP}{dt} = k(800 - P)$

a) $\int \frac{1}{800-P}\,dP = \int k\,dt$

$-\ln|800 - P| = kt + C_1$

$\ln\frac{1}{800-P} = kt + C_1$

$\frac{1}{800-P} = Ce^{kt}$

$\frac{1}{800-500} = Ce^{k(0)}$

$C = \frac{1}{300}$

$\frac{1}{800-P} = \frac{1}{300}e^{kt}$

$800 - P = \frac{300}{e^{kt}} = 300e^{-kt}$

$$\boxed{P = 800 - 300e^{-kt}}$$

24. $\frac{dH}{dt} = -k(H-200)$

a) $\int \frac{1}{H-200}\,dH = \int -k\,dt$

$\ln|H-200| = -kt + C_1$

$|H-200| = C_2 e^{-kt}$

$H - 200 = Ce^{-kt}$

$H(0) = 20°C$

$20 - 200 = Ce^{-k(0)}$

$C = -180$

$\boxed{H = 200 - 180e^{-kt}}$

b) $120 = 200 - 180e^{-30k}$

$\boxed{k = 0.02703}$

25. $T(t) = T_s + (T_0 - T_s)e^{-kt} \Leftrightarrow \frac{dT}{dt} = -k(T - T_s)$

a) $\frac{dT}{dt} = -k(T - 68°)$

b) $\int \frac{1}{T-68}\,dT = \int -k\,dt$

$\ln|T-68| = -kt + C_1$

$T - 68 = Ce^{-kt}$

$T = 68 + Ce^{-kt}$

$T(0) = 68 + Ce^{-k(0)} = 90.3 \Rightarrow C = 90.3 - 68 = 22.3$

$T(1) = 68 + 22.3e^{-k} = 89 \Rightarrow k = .0601$

$T = 68 + 22.3e^{-.0601t}$

$98.6 = 68 + 22.3e^{-.0601t} \Rightarrow t = -5.265$ hrs (prior to 9 AM)

= 5 hrs and 16 min < 9 AM

= $\boxed{3{:}44 \text{ AM}}$

26. $\frac{dy}{dx} = \frac{\cos x}{3y^2}$ ; $y(\pi)=5$

$\int 3y^2 dy = \int \cos x \, dx$

$y^3 = \sin x + C$

$5^3 = \sin \pi + C \Rightarrow C = 125$

$\boxed{y = \sqrt[3]{\sin x + 125}}$

27. $\frac{dy}{dt} = 1500 e^{3t/4}$ ; $y(0) = 2000$

$\int dy = \int 1500 e^{3/4 t} dt$

$y = \left(\frac{4}{3}\right) 1500 e^{3/4 t} + C$

$2000 = 2000 e^{3/4(0)} + C \Rightarrow C = 0$

$y = 2000 e^{3/4 t}$

$y(4) = 2000 e^3 = \boxed{40171.074}$

28. $\frac{dN}{dt} = 2N$ ; $N(0) = 3$

$\int \frac{1}{2N} dN = \int dt$

$\frac{1}{2} \ln|2N| = t + C_1$

$\ln|2N| = 2t + C_2$

$|2N| = e^{2t + C_2}$

$2N = C_3 e^{2t}$

$N = C e^{2t}$

$3 = C e^0$

$N = 3e^{2t}$

$1210 = 3e^{2t}$

$t = 2.9999$

$\boxed{t = 3}$

29. $\frac{dW}{dt} = kW$ ; $W(0)=2$ ; $W(4)=3.5$ Find $W(6)$

$\int \frac{1}{W}\, dW = \int k\, dt$

$\ln|W| = kt + C_1$

$W = Ce^{kt} \Rightarrow C=2$

$W = 2e^{kt}$

$3.5 = 2e^{4k} \Rightarrow k = .1399$

$W = 2e^{.1399t}$

$W(6) = \boxed{4.630 \text{ lbs}}$

30. $\boxed{\frac{ds}{dt} = k\sqrt{\frac{1}{6}s}}$

31. $\frac{dy}{dt} = ky$

$y = 1500e^{kt}$

$6000 = 1500e^{2k} \Rightarrow k = 0.6931$

$y = 1500e^{0.6931t}$

$y(3) = 11998.302$

$\frac{11998}{1500} = \boxed{8}$